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NOTES ON

FOUNDATIONS OF CONTINUUM

MECHANICS

for

Mechanics of Deformable Bodies

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by

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1.1 Scalars and Vectors

Once a suitable system of physical units is selected, certain physical quantities can be placed into one-to-one correspondence with the real numbers. Quantities of this kind are called scalars. The length l of a bar, the mass m of a piece of metal, and the distance d between two points in space are familiar examples of scalar quantities. There are other physical quantities, however, such as force, state of stress in a deformed solid, etc., whose mathematical description requires more than the field of real numbers. A subclass of such quantities comprises those physical or geometrical quantities which can be represented by vectors, i. e., directed line elements in space. The magnitude of the vector quantity is represented by the length of the line segment whose direction indicates the direction of the vector quantity. With this definition, two line segments of equal lengths that are parallel and have the same directions represent the same vector. We use bold-faced letters, such as \underline{A} , \underline{B} , and \underline{C} , to designate vectors and denote their magnitudes by A , B , and C , respectively. Examples of vector quantities are force, velocity, and acceleration, in particle dynamics

If \underline{A} is a vector and m a real number, then $m\underline{A}$ is a new vector that is collinear with \underline{A} and has a length $|m|A$, where $|m|$ denotes the absolute value of m . If m is negative, then the direction of the new vector $m\underline{A}$ is opposite to that of \underline{A} . The following vector operations should be familiar to the reader:

a) Vector Summation. The sum \underline{C} of two vectors \underline{A} and \underline{B} , $\underline{C} = \underline{A} + \underline{B}$, is defined by the parallelogram law, see Fig. 1.1. Note

that, for two vectors \underline{E} and \underline{F} , the difference \underline{D} is defined by $\underline{D} = \underline{E} - \underline{F} = \underline{E} + (-\underline{F})$, and that the vector addition is commutative, i. e., $\underline{A} + \underline{B} = \underline{B} + \underline{A}$, and associative, i. e., $(\underline{A} + \underline{B}) + \underline{C} = \underline{A} + (\underline{B} + \underline{C})$.

b) Scalar Multiplication. Consider two vectors \underline{A} and \underline{B} . The scalar (or dot) product of \underline{A} and \underline{B} , denoted by $\underline{A} \cdot \underline{B}$, is a real number c obtained by multiplying the magnitude of \underline{A} , namely A , by the orthogonal projection of \underline{B} on \underline{A} , i. e., $c = \underline{A} \cdot \underline{B} = AB \cos \theta$, where θ is the angle formed by the directions of \underline{A} and \underline{B} (see Fig. 1.2). Note that the scalar product is commutative, i. e., $\underline{A} \cdot \underline{B} = \underline{B} \cdot \underline{A}$, and distributive, i. e., $\underline{A} \cdot (\underline{B} + \underline{C}) = \underline{A} \cdot \underline{B} + \underline{A} \cdot \underline{C}$. If the directions of \underline{A} and \underline{B} are orthogonal, then $\underline{A} \cdot \underline{B} = 0$.

c) Vector Multiplication. The vector product of two non-parallel vectors \underline{A} and \underline{B} is a vector \underline{C} whose direction is normal to the directions of both \underline{A} and \underline{B} , the triad \underline{A} , \underline{B} , \underline{C} , being right-handed, and whose magnitude is given by $AB \sin \theta$ (see Fig. 1.3). We denote the vector product by the symbol \times , i. e., we write $\underline{A} \times \underline{B} = \underline{C}$, and note that $\underline{A} \times \underline{B} = -\underline{B} \times \underline{A}$, and $\underline{A} \times (\underline{B} + \underline{D}) = \underline{A} \times \underline{B} + \underline{A} \times \underline{D}$. For any two non-zero vectors, $\underline{A} \neq \underline{0}$, $\underline{B} \neq \underline{0}$, the relation $\underline{A} \times \underline{B} = \underline{0}$ is valid if and only if \underline{A} and \underline{B} are parallel, where $\underline{0}$ denotes the null vector, a vector with zero magnitude and arbitrary direction.

It is important to note that the vector operations (a), (b), and (c) above are defined without reference to a coordinate system.

1.2 Coordinate System and Index Notation

In a three-dimensional Euclidean space, a vector may be defined by three numbers which are called its components with respect to a system

of coordinates. Consider a right-handed system of rectangular Cartesian coordinates x_1, x_2, x_3 . A point P can be represented by its position vector \underline{x} , that is, the directed line segment OP . Let $\underline{e}_1, \underline{e}_2$, and \underline{e}_3 be three base vectors, that is, vectors of unit length in the positive coordinate directions. By the definitions of vector summation and scalar multiplication we have (see Fig. 2.1):

$$\underline{x} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + x_3 \underline{e}_3 = \sum_{i=1}^3 x_i \underline{e}_i \quad (2.1)$$

where the components of \underline{x} are

$$x_1 = \underline{x} \cdot \underline{e}_1, \quad x_2 = \underline{x} \cdot \underline{e}_2, \quad x_3 = \underline{x} \cdot \underline{e}_3$$

Accordingly, (2.1) may be written in the form

$$\underline{x} = \sum_{i=1}^3 (\underline{x} \cdot \underline{e}_i) \underline{e}_i \quad (2.2)$$

Equations (2.1) and (2.2) are more compactly written using the following summation convention, which was introduced by Einstein:

Summation Convention. A repeated subscript in a monomial stands for the sum of the three terms obtained by successively giving to this subscript the values 1, 2, and 3. Note that, for this rule to be meaningful, a subscript can at most occur twice in each monomial. Using this rule, we may rewrite (2.1) and (2.2) as follows:

$$\underline{x} = x_i \underline{e}_i = (\underline{x} \cdot \underline{e}_i) \underline{e}_i \quad ; \quad i = 1, 2, 3 \quad (2.3)$$

Let A_i and B_i denote the components of vectors \underline{A} and \underline{B} , respectively, i. e., $\underline{A} = A_i \underline{e}_i$, and $\underline{B} = B_j \underline{e}_j$; $i, j = 1, 2, 3$. The dot product of \underline{A} and \underline{B} then is

$$\begin{aligned} \underline{A} \cdot \underline{B} &= (A_i \underline{e}_i) \cdot (B_j \underline{e}_j) \\ &= (A_i B_j) \underline{e}_i \cdot \underline{e}_j \quad ; \quad i, j = 1, 2, 3 \end{aligned} \quad (2.4)$$

From the definition of unit base vectors, we have

$$\underline{e}_i \cdot \underline{e}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} .$$

It is convenient to introduce a new symbol δ_{ij} , called the Kronecker delta, defined by

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} . \quad (2.5)$$

Equation (2.4) then becomes

$$\begin{aligned} \underline{A} \cdot \underline{B} &= (A_i B_j) \delta_{ij} = A_i B_i \\ &= A_1 B_1 + A_2 B_2 + A_3 B_3 \end{aligned} \quad (2.6)$$

The magnitude A of the vector \underline{A} is given by

$$A^2 = A_i A_i \quad (2.7)$$

The cross product of \underline{A} and \underline{B} is

$$\begin{aligned} \underline{A} \times \underline{B} &= (A_i \underline{e}_i) \times (B_j \underline{e}_j) \\ &= (A_i B_j) \underline{e}_i \times \underline{e}_j \end{aligned} \quad (2.8)$$

From the definitions of the cross product and the unit base vectors we have

$$\begin{aligned}
\tilde{e}_1 \times \tilde{e}_2 &= \tilde{e}_3, & \tilde{e}_2 \times \tilde{e}_3 &= \tilde{e}_1, & \tilde{e}_3 \times \tilde{e}_1 &= \tilde{e}_2 \\
\tilde{e}_2 \times \tilde{e}_1 &= -\tilde{e}_3, & \tilde{e}_3 \times \tilde{e}_2 &= -\tilde{e}_1, & \tilde{e}_1 \times \tilde{e}_3 &= -\tilde{e}_2.
\end{aligned}
\tag{2.9}$$

It is convenient to introduce a new symbol e_{ijk} , called the permutation symbol, defined as follows:

$$e_{ijk} = \begin{cases} +1 \\ -1 \\ 0 \end{cases} \begin{cases} \text{according to} \\ \text{whether } i, j, k \end{cases} \begin{cases} \text{form an even} \\ \text{form an odd} \\ \text{do not form a} \end{cases} \begin{cases} \text{permutation} \\ \text{of } 1, 2, 3 \end{cases}. \tag{2.10}$$

Note that according to this definition the relations (2.9) may be concisely written as follows:

$$\tilde{e}_i \times \tilde{e}_j = e_{ijk} \tilde{e}_k. \tag{2.9a}$$

Accordingly, (2.8) yields

$$\begin{aligned}
\tilde{A} \times \tilde{B} &= (A_i B_j) e_{ijk} \tilde{e}_k \\
&= \tilde{e}_1 (A_2 B_3 - A_3 B_2) + \tilde{e}_2 (A_3 B_1 - A_1 B_3) + \tilde{e}_3 (A_1 B_2 - A_2 B_1).
\end{aligned}
\tag{2.11}$$

If $\tilde{C} = C_k \tilde{e}_k = \tilde{A} \times \tilde{B}$, it follows from (2.11) that

$$C_k = A_i B_j e_{ijk} = e_{kij} A_i B_j. \tag{2.12}$$

The following relations are direct consequences of the summation convention and the definitions (2.5) and (2.10):

$$\begin{aligned}
e_{ijk} e_{ijk} &= 6 \\
e_{ijk} e_{ijl} &= 2 \delta_{kl} \\
e_{ijk} e_{ilm} &= \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}
\end{aligned}
\tag{2.13}$$

The repeated subscripts are called dummy subscripts. A dummy subscript may be replaced by any other subscript letter which is not otherwise used in the same relation. The subscripts that occur only once in each monomial of an equation are called live subscripts. The first equation in (2.13), for example, contains only dummy subscripts, while in the second equation l and k are both live subscripts and i is a dummy subscript.

1.3 Coordinate Transformations

Consider now the transformation of coordinates. Let x_1, x_2, x_3 and x'_1, x'_2, x'_3 be the coordinates of one and the same point P with the position vector \underline{x} in two systems of right-handed rectangular Cartesian coordinates that have the same origin O . Let \underline{e}_i and $\underline{e}'_i, i = 1, 2, 3$, denote the base vectors of the unprimed and primed coordinates, respectively. The vector \underline{x} may be expressed as

$$\underline{x} = x_i \underline{e}_i = x'_i \underline{e}'_i \quad ; \quad i = 1, 2, 3 \quad . \quad (3.1)$$

Taking the dot product of this equation first with \underline{e}_j and then with \underline{e}'_j , we obtain

$$x_j = (\underline{e}'_i \cdot \underline{e}_j) x'_i \quad , \quad (3.2)$$

$$x'_j = (\underline{e}_j \cdot \underline{e}_i) x_i \quad ; \quad i, j = 1, 2, 3 \quad . \quad (3.3)$$

Since \underline{e}'_i and \underline{e}_j are unit vectors, $(\underline{e}'_i \cdot \underline{e}_j)$ denotes the cosine of the angle formed by the x'_i and x_j axes. Denoting the cosine of this angle by c_{ij} , i.e., $c_{ij} = \underline{e}'_i \cdot \underline{e}_j$, we rewrite (3.2) and (3.3) as

$$x_j = c_{ij} x'_i \quad , \quad (3.4)$$

$$x'_j = c_{ji} x_i \quad ; \quad i, j = 1, 2, 3 \quad . \quad (3.5)$$

Note that, in accord with the summation convention, for fixed j , the terms on the right side of (3.4) and (3.5) are to be summed on i , for $i = 1, 2, 3$.

Substitution of (3.4) into (3.5) yields

$$x'_j = c_{ji} c_{ki} x'_k$$

which must be an identity, leading to

$$c_{ji} c_{ki} = \delta_{jk} \quad . \quad (3.6)$$

Similarly, substitution of (3.5) into (3.4) results in the following relation:

$$c_{ij} c_{ik} = \delta_{jk} \quad . \quad (3.7)$$

Consider a vector $\underline{\tilde{A}}$ with components A_i and A'_i in the unprimed and primed coordinate systems, respectively, i. e.,

$$\underline{\tilde{A}} = A_i \underline{\tilde{e}}_i = A'_i \underline{\tilde{e}}'_i \quad . \quad (3.8)$$

Taking the dot product of both sides of (3.8) first with $\underline{\tilde{e}}_j$ and then with $\underline{\tilde{e}}'_j$, we obtain

$$A_j = c_{ij} A'_i \quad , \quad (3.9)$$

$$A'_j = c_{ji} A_i \quad . \quad (3.10)$$

With reference to the unprimed coordinate system, the vector \underline{A} is specified by its components A_i . These components change to A'_i as a new (primed) coordinate system is introduced. The rule (3.10) links the components of \underline{A} in the two coordinate systems. Therefore, a vector may be also defined by a triple of components A_i which transform in accordance with the rule (3.10) under the coordinate transform (3.5). It is customary to refer to vectors as tensors of order 1.

1.4 Scalar and Vector Fields

Let a scalar function $\varphi = \varphi(\underline{x})$ be defined and continuous in a finite region R of the three-dimensional Euclidean space whose points are referenced by means of a rectangular Cartesian coordinate system. Consider a generic point x_i^0 in R and denote the value of φ at this point by φ^0 . The locus of the points in R at which the scalar function $\varphi(\underline{x})$ attains the value of φ^0 is a surface, called the level surface, with the following equation:

$$\varphi(\underline{x}) = \varphi^0 = \text{constant} \quad . \quad (4.1)$$

The rate of change of the scalar function φ in the direction of the x_1 -axis at point x_i^0 is given by $[\frac{\partial \varphi}{\partial x_1}]_{x_i = x_i^0}$. Thus the quantity $\frac{\partial \varphi}{\partial x_1}$ is a new scalar field which defines the rate of change of the field φ in the direction of the x_1 -axis in R . Similarly, $\frac{\partial \varphi}{\partial x_2}$ and $\frac{\partial \varphi}{\partial x_3}$ are the scalar fields describing the rate of change of the scalar φ in the x_2 - and x_3 -directions, respectively. The triple $\frac{\partial \varphi}{\partial x_i}$, $i = 1, 2, 3$, may, therefore,

be viewed as components of a vector field in R , called the gradient of φ . The symbol ∇ , pronounced "del," is commonly used to denote the gradient, i. e.,

$$\nabla \varphi = \left(\frac{\partial \varphi}{\partial x_i} \right) \underline{e}_i ; i = 1, 2, 3 \quad . \quad (4.2a)$$

The operator $\nabla \equiv \underline{e}_i \frac{\partial}{\partial x_i}$ operates on a scalar field $\varphi(\underline{x})$ and yields the vector field (4.2a). The notation $\text{grad } \varphi$ is also employed to denote $\nabla \varphi$.

The rate of change of the scalar φ in any given direction is now defined by the projection of $\nabla \varphi$ into that direction. Let $\underline{\mu}$ be a unit vector with components μ_i . The gradient of φ in the direction of $\underline{\mu}$ at point \underline{x}_i^0 is

$$\varphi^{(\mu)}(\underline{x}^0) = [\underline{\mu} \cdot \nabla \varphi]_{\underline{x}=\underline{x}^0} = \left[\mu_i \frac{\partial \varphi}{\partial x_i} \right]_{\underline{x}=\underline{x}^0}$$

If $\underline{\mu}$ is tangent to the level surface $\varphi = \varphi^0$, then $\varphi^{(\mu)}(\underline{x}^0)$ is zero. Thus $\nabla \varphi$ is normal to the level surface $\varphi = \varphi^0$. Moreover, $\varphi^{(\mu)}(\underline{x}^0)$ attains its maximum value when $\underline{\mu}$ and $\nabla \varphi$ are collinear and possess the same direction. Thus, at each point in R , $\nabla \varphi$ points toward the direction of maximum rate of increase of the scalar φ , and is in magnitude equal to this rate.

Introducing the following notation for partial differentiation:

$$\frac{\partial \varphi}{\partial x_i} \equiv \varphi_{,i} \quad , \quad (4.3a)$$

where a comma followed by subscript letter is to be interpreted as partial differentiation with respect to the corresponding coordinate, we may write (4.2a) as

$$\underline{\nabla} \varphi = \underline{e}_i \varphi_{,i} \quad ; \quad i = 1, 2, 3 \quad . \quad (4.2b)$$

Note that the introduction of this notation is purely a matter of convenience. Some authors use the notation

$$\frac{\partial \varphi}{\partial x_i} \equiv \partial_i \varphi \quad (4.3b)$$

which has certain advantages. Here we shall use a comma followed by a subscript to denote partial differentiation with respect to the respective coordinate.

Relations (4.2) suggest that we may view the "del operator," $\underline{\nabla} \equiv \underline{e}_i \frac{\partial}{\partial x_i}$, as a vector and form its dot and cross products with vector fields which are continuous. Let $\underline{v} = v_i \underline{e}_i$ be a vector function of \underline{x} in a finite region R . Assume that the components $v_i = v_i(\underline{x})$ are continuous functions of their arguments in R . The dot product of $\underline{\nabla}$ with \underline{v} is

$$\begin{aligned} \underline{\nabla} \cdot \underline{v} &= \left(\frac{\partial}{\partial x_i} \underline{e}_i \right) \cdot (v_j \underline{e}_j) \\ &= \left(\frac{\partial v_j}{\partial x_i} \right) \underline{e}_i \cdot \underline{e}_j \\ &= v_{j,i} \delta_{ij} = v_{i,i} \\ &= v_{1,1} + v_{2,2} + v_{3,3} \end{aligned} \quad (4.4a)$$

which is called divergence of \underline{v} , and is usually written as

$$\text{div } \underline{v} \equiv v_{i,i} \equiv \underline{\nabla} \cdot \underline{v} \quad . \quad (4.4b)$$

Similarly, we may consider the cross product of $\underline{\nabla}$ and \underline{v} ,

$$\begin{aligned} \underline{\nabla} \times \underline{v} &= \left(\frac{\partial}{\partial x_i} \underline{e}_i \right) \times (v_j \underline{e}_j) \\ &= \left(\frac{\partial v_j}{\partial x_i} \right) \underline{e}_i \times \underline{e}_j \\ &= (v_{j,i}) e_{ijk} \underline{e}_k \end{aligned}$$

$$\begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \underline{e}_1 (v_{3,2} - v_{2,3}) + \underline{e}_2 (v_{1,3} - v_{3,1}) + \underline{e}_3 (v_{2,1} - v_{1,2}) \quad (4.5a)$$

which is called curl of \underline{v} , and is usually denoted by

$$\text{curl } \underline{v} \equiv \underline{\nabla} \times \underline{v} \quad . \quad (4.5b)$$

Note that, according to (4.3b), (4.5a) is written as

$$\underline{\nabla} \times \underline{v} = (\partial_i v_j) e_{ijk} \underline{e}_k \quad (4.5c)$$

which possesses the symmetry of usual cross product of vectors (compare with Eq. (2.12)).

Forming the dot product of $\underline{\nabla}$ with itself, we obtain

$$\begin{aligned}
 \underline{\nabla} \cdot \underline{\nabla} &\equiv \left(\frac{\partial^2}{\partial x_i \partial x_j} \right) \underline{e}_i \cdot \underline{e}_j \\
 &\equiv \frac{\partial^2}{\partial x_i \partial x_j} \delta_{ij} \equiv \frac{\partial^2}{\partial x_i \partial x_i} \\
 &\equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \\
 &\equiv \nabla^2 \equiv \Delta
 \end{aligned}
 \tag{4.6}$$

which is the usual Laplace's operator. As an exercise, the reader may verify the following identities:

$$\begin{aligned}
 \text{div grad } \varphi &= \nabla^2 \varphi \quad , \\
 \text{div curl } \underline{v} &= 0 \quad , \\
 \text{curl curl } \underline{v} &= \text{grad div } \underline{v} - \nabla^2 \underline{v} \quad ,
 \end{aligned}$$

where φ and \underline{v} are twice differentiable scalar and vector fields, respectively.

Since at each point in R the quantity $\underline{\nabla} \varphi$ defines a vector, its components $\varphi_{,i}$ must transform according to the law (3.9). From the chain rule of differentiation, we have

$$\frac{\partial \varphi}{\partial x_i} = \frac{\partial \varphi}{\partial x'_j} \frac{\partial x'_j}{\partial x_i} = \frac{\partial \varphi}{\partial x'_j} c_{ji} \quad ,$$

where $\partial x'_j / \partial x_i = c_{ji}$ by Eq. (3.4).

1.5 Tensors

In Section 1, a scalar quantity was associated with a real number, while a vector quantity was represented by an oriented line segment. There are other physical and geometrical quantities such as stress and strain fields in a deformed continuum, which cannot be represented by real numbers or be vectors, that is, more is needed for their mathematical description. Quantities of this kind are called tensors.

Let \underline{u} and \underline{v} be two independent vectors, and consider a quantity \underline{w} defined by

$$\underline{w} = \underline{u} \underline{v} \quad . \quad (5.1a)$$

Note that \underline{w} is neither the dot product, nor the cross product of \underline{u} and \underline{v} . It is simply defined by a pair of vectors \underline{u} and \underline{v} that are ordered as in (5.1a). For example, \underline{u} may be the del operator $\underline{\nabla} \equiv \underline{e}_i \frac{\partial}{\partial x_i}$, in which case \underline{w} is the gradient of the vector field $\underline{v} = \underline{e}_j v_j(x)$, i. e., $\underline{w} = \underline{\nabla} \underline{v} = (v_{i,j}) \underline{e}_j \underline{e}_i$. Note that (5.1a) defines a definite order for \underline{u} and \underline{v} . Thus, in general, $\underline{u} \underline{v} \neq \underline{v} \underline{u}$. In the present example, $\underline{v} \underline{\nabla}$ is not, as yet, defined.

Referring to a rectangular Cartesian coordinate system x_i , we let u_i and v_i , $i = 1, 2, 3$, denote the respective components of \underline{u} and \underline{v} . Relation (5.1a) may then be expressed as

$$\begin{aligned} \underline{w} &= (u_i \underline{e}_i)(v_j \underline{e}_j) \\ &= (u_i v_j) \underline{e}_i \underline{e}_j \quad , \quad i, j = 1, 2, 3 \quad , \end{aligned} \quad (5.1b)$$

where, as usual, the summation convention is implied. If \underline{b} is to represent a physical quantity, it should be invariant under all coordinate transformations; in particular, under the coordinate transformation (3.5). We thus must have

$$\underline{b} = (u_i \ v_j) \underline{e}_i \underline{e}_j = (u'_i \ v'_j) \underline{e}'_i \underline{e}'_j \quad , \quad (5.2)$$

where the primed quantities refer to the primed coordinate system. Taking the dot products of (5.2) with \underline{e}'_k and with \underline{e}'_l , consecutively, we obtain

$$u'_k \ v'_l = c_{ki} \ c_{lj} \ u_i \ v_j \quad , \quad i, j, k, l = 1, 2, 3 \quad , \quad (5.3a)$$

which defines the law of transformation of the components of \underline{b} under the coordinate transformation (3.5). Similarly, dotting both sides of (5.2) with \underline{e}_k and \underline{e}_l , consecutively, we arrive at

$$u_k \ v_l = c_{ik} \ c_{jl} \ u'_i \ v'_j \quad . \quad (5.3b)$$

Hence, for \underline{b} to represent a physical quantity which exists independently of a particular rectangular Cartesian coordinate system that is used for the description of its components, these components must transform according to (5.3) under the coordinate transformation (3.5). For $i, j = 1, 2, 3$, the nine scalar quantities

$$\begin{array}{ccc} (u_1 \ v_1) & (u_1 \ v_2) & (u_1 \ v_3) \\ (u_2 \ v_1) & (u_2 \ v_2) & (u_2 \ v_3) \\ (u_3 \ v_1) & (u_3 \ v_2) & (u_3 \ v_3) \end{array}$$

define the components of \underline{w} in the x_i coordinate system. Introducing the following notation

$$\underline{w} = w_{ij} \underline{e}_i \underline{e}_j \quad (5.1c)$$

where

$$w_{ij} \equiv u_i v_j ,$$

we may speak of \underline{w} as a second order tensor with components w_{ij} in the x_i coordinate system. Thus any physical quantity which can be described in a given rectangular Cartesian coordinate system x_i by its nine scalar components, say w_{ij} , which transform according to the law.

$$w'_{ij} = c_{ik} c_{jl} w_{kl} \quad (5.3c)$$

under the coordinate transformation (3.5), is called a second order tensor.

From this terminology, it is clear why vectors are called tensors of first order. Moreover, a slight amount of imagination would reveal that scalars do really deserve the zeroth order of tensor-ship; they are called tensors of order zero.

By dotting both sides of (5.1c) with \underline{e}_k and with \underline{e}_l , we obtain

$$w_{lk} = \underline{e}_l \cdot \underline{w} \cdot \underline{e}_k \quad (5.4)$$

which formally defines the components of the second order tensor \underline{w} in the rectangular Cartesian coordinate system x_i .

With the above preliminaries, we shall now generalize the definitions of tensors of order zero (scalars) and one (vectors) to tensors

of higher order. A tensor of order 2 is a geometrical or physical quantity which may be specified by its 3^2 components with respect to a given coordinate system. Let \underline{T} be a second order tensor with components T_{ij} and T'_{ij} in the unprimed and primed coordinate systems, respectively, i. e., let

$$\underline{T} = T_{ij} \underline{e}_i \underline{e}_j = T'_{ij} \underline{e}'_i \underline{e}'_j \quad (5.5)$$

Taking the dot product of (5.5) first with \underline{e}_k and then with \underline{e}_l , we obtain

$$T_{kl} = c_{ik} c_{jl} T'_{ij} \quad (5.6)$$

Similarly, the dot product of (5.5) with \underline{e}'_k and \underline{e}'_l yields

$$T'_{kl} = c_{ki} c_{lj} T_{ij} \quad (5.7)$$

Equations (5.6) and (5.7) are now used to define a tensor of second order as follows: with reference to the coordinate system x_i , the 3^2 components T_{ij} define a second order tensor if these components transform according to (5.7) under the coordinate transformation (3.5). Similarly, a tensor of order 3 is specified by 3^3 components T_{ijk} that transform according to

$$T'_{ijk} = c_{im} c_{jn} c_{kp} T_{mnp} \quad (5.8)$$

In general, a tensor of order n may be expressed as

$$\begin{aligned} \underline{T} &= T_{i_1 i_2 \dots i_n} \underline{e}_{i_1} \underline{e}_{i_2} \dots \underline{e}_{i_n} \\ &= T'_{i_1 i_2 \dots i_n} \underline{e}'_{i_1} \underline{e}'_{i_2} \dots \underline{e}'_{i_n}, \quad i_1, i_2, \dots, i_n = 1, 2, 3, \end{aligned} \quad (5.9)$$

where the 3^n components $T_{j_1 j_2 \dots j_n}$ in the unprimed coordinates are linked with $T'_{i_1 i_2 \dots i_n}$ in the primed coordinates by the following transformation:

$$T'_{i_1 i_2 \dots i_n} = c_{i_1 j_1} c_{i_2 j_2} \dots c_{i_n j_n} T_{j_1 j_2 \dots j_n} ;$$

$$i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n = 1, 2, 3 \quad . \quad (5.10)$$

Let $\underline{v} = v_i \underline{e}_i$ be a vector and $\underline{\mathfrak{T}} = T_{ij} \underline{e}_i \underline{e}_j$ a second order tensor. The dot product of \underline{v} and $\underline{\mathfrak{T}}$ is defined as

$$\begin{aligned} \underline{v} \cdot \underline{\mathfrak{T}} &= v_i (\underline{e}_i \cdot \underline{e}_j) \underline{e}_k T_{jk} \\ &= (v_i T_{jk}) (\underline{e}_i \cdot \underline{e}_j) \underline{e}_k \\ &= (v_i T_{jk}) \delta_{ij} \underline{e}_k = v_j T_{jk} \underline{e}_k \end{aligned} \quad (5.11)$$

(continued on next page)

which is a vector. The dot product of $\underline{\underline{T}}$ with $\underline{\underline{v}}$, on the other hand, is defined as

$$\underline{\underline{T}} \cdot \underline{\underline{v}} = T_{jk} \underline{\underline{e}}_j (\underline{\underline{e}}_k \cdot \underline{\underline{e}}_i) v_i = T_{jk} v_i \delta_{ki} \underline{\underline{e}}_j = T_{jk} v_k \underline{\underline{e}}_j \quad (5.12)$$

which also is a vector. Note that (5.11) and (5.12) represent different vectors unless $\underline{\underline{T}}$ is a symmetric tensor, that is, unless $T_{ij} \underline{\underline{e}}_i \underline{\underline{e}}_j$ is equal to its transpose $\underline{\underline{T}}^T = T_{ji} \underline{\underline{e}}_i \underline{\underline{e}}_j$. Since $\underline{\underline{T}} \cdot \underline{\underline{v}}$ in (5.12) is a vector, say $\underline{\underline{u}}$, we may write

$$u_i = T_{ij} v_j \quad ; \quad i, j = 1, 2, 3 \quad (5.13)$$

This equation expresses a linear transformation in the three-dimensional Euclidean space; a vector $\underline{\underline{v}}$ in this space is transformed into another vector $\underline{\underline{u}}$ in the same space. Such a transformation is called orthogonal if $\underline{\underline{T}}$ is an orthogonal tensor, that is, if

$$T_{ij} T_{kj} = \delta_{ik} \quad (5.14)$$

The tensor $\underline{\underline{c}} = c_{ij} \underline{\underline{e}}_i \underline{\underline{e}}_j$ which satisfies condition (3.7) is an orthogonal tensor. In this context, Eq. (3.10) may be interpreted as rotation of a vector $\underline{\underline{A}}$ with components A_i into another vector $\underline{\underline{A}}'$ with components A'_j when these vectors are referred to one and the same coordinate system x_i . An orthogonal tensor that transforms a vector $\underline{\underline{A}}$ into itself is called the unit tensor. From the definition of the Kronecker delta, we have $A_i = \delta_{ij} A_j$ and, therefore,

$$\underline{\underline{I}} = \delta_{ij} \underline{\underline{e}}_i \underline{\underline{e}}_j = \underline{\underline{e}}_i \underline{\underline{e}}_i \quad (5.15)$$

is the unit tensor.

The tensor product of a second order tensor $\underline{\underline{J}}$ and a vector \underline{v} is a tensor of order 3, defined as $\underline{\underline{J}}\underline{v} = T_{ij} v_k \underline{e}_i \underline{e}_j \underline{e}_k$. Setting $j = k$ in the expression $T_{ij} v_k$, we obtain the dot product of $\underline{\underline{J}}$ and \underline{v} , namely \underline{u} . This is called contraction. In general, making two letter indices of the components of a tensor of order p identical results in the components of a tensor of order $p-2$. Contraction of the indices of a second order tensor T_{ij} results in a scalar which is called the trace of $\underline{\underline{J}}$ and is denoted by $\text{tr } \underline{\underline{J}} = T_{ii}$.

As was mentioned before, a second order tensor is called symmetric if it is equal to its transpose. In this connection, a tensor $\underline{\underline{J}}$ is called antisymmetric if we have $\underline{\underline{J}} = -\underline{\underline{J}}^T$. In general, any second order tensor $\underline{\underline{J}}$ can be written as a sum of two parts, a symmetric and an antisymmetric part, i. e.,

$$\begin{aligned} T_{ij} &= \frac{1}{2} (T_{ij} + T_{ji}) + \frac{1}{2} (T_{ij} - T_{ji}) \\ &= T_{(ij)} + T_{[ij]} \end{aligned} \quad (5.16)$$

where the components of the symmetric and antisymmetric parts of $\underline{\underline{J}}$ are denoted by $T_{(ij)}$ and $T_{[ij]}$, respectively. The dual vector of a non-symmetric second order tensor is defined by

$$\begin{aligned} \underline{t} &= t_k \underline{e}_k = T_{ij} \underline{e}_i \times \underline{e}_j \\ &= e_{ijk} T_{ij} \underline{e}_k \end{aligned} \quad (5.17)$$

or

$$t_k = e_{ijk} T_{ij} \quad (5.18)$$

Note that the dual vector of a symmetric tensor is the zero vector, since, by definition (2.10), e_{ijk} in (5.18) is antisymmetric with respect to the exchange of i and j . Solving (5.18) for T_{ij} , we obtain

$$T_{ij} = \frac{1}{2} e_{ijk} t_k \quad (5.19)$$

A tensor of second order may be viewed as a vector in a 9-dimensional Euclidean space. Let $\underline{T} = T_{ij} \underline{e}_i \underline{e}_j$ and $\underline{V} = V_{ij} \underline{e}_i \underline{e}_j$ be tensors of second order that are connected through a tensor of fourth order $\underline{B} = B_{ijkl} \underline{e}_i \underline{e}_j \underline{e}_k \underline{e}_l$ as follows:

$$T_{ij} = B_{ijkl} V_{lk} \quad (5.20)$$

or in the invariant notation

$$\underline{T} = \underline{B} : \underline{V} \quad (5.21)$$

where the double dot designates two contractions. Equation (5.20) defines a linear transformation of a vector in 9-dimensional Euclidean space to another vector in the same space.

Since we have viewed a tensor as an invariant quantity that exists independently of any coordinate system which may be used to represent its components, the generalization of our results to more general coordinate systems would only require a proper change of the base vectors. We shall consider this later.

1.6 Matrix and Determinant

Referred to a rectangular Cartesian coordinate system, the components T_{ij} ; $i, j = 1, 2, 3$, of a second order tensor \underline{T} may be represented collectively in a form called matrix as follows:

$$\tilde{T} \equiv [T_{ij}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}. \quad (6.1)$$

Note that the first subscript letter in the element T_{ij} of the matrix $\tilde{T} \equiv [T_{ij}]$ denotes the row-location, and the second subscript letter the column-location of this element in the single entity \tilde{T} . Note also that a definite coordinate system is implied when matrices are used to represent tensors. We use the notation \tilde{T} to denote the matrix representation of the tensor \mathfrak{T} with respect to the rectangular Cartesian coordinate system x_i .

A vector $\mathfrak{y} = v_i \mathfrak{e}_i$ can be represented by a column matrix;

$$\tilde{v} \equiv \{v_i\} \equiv \left\{ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} \right\} \quad (6.2a)$$

whose transpose, \tilde{v}^T , is a row vector

$$\tilde{v}^T = \{v_i\}^T = \{v_1 \ v_2 \ v_3\}. \quad (6.2b)$$

If $\tilde{A} \equiv [A_{ij}]$ and $\tilde{B} \equiv [B_{ij}]$, $i, j = 1, 2, 3$, are 3×3 matrices, then the sum $\tilde{C} \equiv [C_{ij}]$ and product $\tilde{D} \equiv [D_{ij}]$ of these matrices are defined by

$$\tilde{C} \equiv [C_{ij}] = [A_{ij} + B_{ij}],$$

and

$$\begin{aligned} \tilde{D} &= [D_{ik}] = \tilde{A} \tilde{B} = [A_{ij}] [B_{jk}] = \\ &= [A_{ij} \ B_{jk}] = \\ &= [A_{i1} \ B_{1k} + A_{i2} \ B_{2k} + A_{i3} \ B_{3k}], \end{aligned} \quad (6.3)$$

respectively. Note that, for an orthogonal matrix \tilde{Q} , we have

$$\tilde{Q} \tilde{Q}^T = \tilde{\delta} ,$$

where $\tilde{\delta}$ is the identity matrix; $\tilde{\delta} \equiv [\delta_{ij}]$.

The transformation (5.13) may now be written in the matrix notation as

$$\tilde{T} \tilde{v} = [T_{ij}] \{v_j\} = \{u_i\} . \quad (6.4)$$

This system of linear equations has a unique solution for $\tilde{v} \equiv \{v_i\}$ if and only if the determinant $\det|T_{ij}|$ of the coefficients of the unknowns v_i , $i = 1, 2, 3$, is nonvanishing. The expansion of the determinant $\det|T_{ij}|$ may be written as

$$\det|T_{ij}| = \frac{1}{6} e_{ijk} e_{rst} T_{ir} T_{js} T_{kt} \quad (6.5)$$

which can be checked by direct expansion. Assuming that $\det|T_{ij}| \neq 0$, equations (6.4) can be solved for \tilde{v} , and one obtains

$$\tilde{v} = \tilde{T}^{-1} \tilde{u} \quad (6.6)$$

where the inverse matrix \tilde{T}^{-1} is a 3×3 matrix which satisfies the following equation:

$$\tilde{T} \tilde{T}^{-1} = \tilde{T}^{-1} \tilde{T} = \tilde{\delta} . \quad (6.7)$$

Thus the elements of \tilde{T}^{-1} are the reduced cofactors of the corresponding elements in \tilde{T} . The cofactor t_{ij} of the element T_{ij} is given by

$$t_{ij} = \frac{1}{2} e_{ipq} e_{jrs} T_{pr} T_{qs} , \quad (6.8)$$

and the reduced cofactor of T_{ij} is obtained by dividing t_{ij} by the $\det|T_{ij}|$. We, therefore, have

$$\tilde{T}^{-1} = \left[\frac{t_{ji}}{\det|T_{pq}|} \right] = \left[\frac{\frac{1}{2} e_{jrs} e_{ipq} T_{pr} T_{qs}}{\frac{1}{6} e_{klm} e_{ghf} T_{kg} T_{lh} T_{mf}} \right] \quad (6.9)$$

which explicitly defines the inverse \tilde{T}^{-1} of the matrix \tilde{T} .

Using the matrix notation, the coordinate transformation (3.5) may be expressed as

$$\{x'_j\} = [c_{ji}] \{x_i\}$$

or

$$\tilde{x}' = \tilde{c} \tilde{x} \quad (6.10)$$

Let us now superpose another transformation, defined by the orthogonal matrix \tilde{c}' , and obtain, from (6.10)

$$\begin{aligned} \tilde{x}'' &= \tilde{c}' \tilde{x}' = \tilde{c}' \tilde{c} \tilde{x} \\ &= \tilde{D} \tilde{x}, \end{aligned} \quad (6.11)$$

where the matrix \tilde{D} is also an orthogonal matrix. Thus two successive rotations of the coordinate system are equivalent to a single rotation defined by $\tilde{D} = \tilde{c}' \tilde{c}$. As an example, consider three successive rotations of the coordinate system that correspond to the

Eulerian angles as follows:

- 1) a rotation through angle φ about x_3 axis ($\tilde{x} \rightarrow \tilde{x}'$),
- 2) a rotation through angle θ about x'_1 axis ($\tilde{x}' \rightarrow \tilde{x}''$), and
- 3) a rotation through angle ψ about x''_3 axis ($\tilde{x}'' \rightarrow \tilde{x}'''$).

We can find a single rotation, defined by $\tilde{x} = \tilde{E} \tilde{x}'''$, which is equivalent to

these three successive rotations. Let $\tilde{x} = \tilde{c}\tilde{x}'$, $\tilde{x}' = \tilde{c}'\tilde{x}''$, and $\tilde{x}'' = \tilde{c}''\tilde{x}'''$. Then $\tilde{x} = \tilde{E}\tilde{x}'''$, where $\tilde{E} = \tilde{c}\tilde{c}'\tilde{c}''$. But we have

$$\tilde{c} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{c}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix},$$

$$\text{and } \tilde{c}'' = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

from which we obtain

$$\tilde{E} = \tilde{c}\tilde{c}'\tilde{c}''$$

$$= \begin{bmatrix} (\cos \psi \cos \varphi - \cos \theta \sin \varphi \sin \psi)(-\sin \psi \cos \varphi - \cos \theta \sin \varphi \cos \psi)(\sin \theta \sin \varphi) \\ (\cos \psi \sin \varphi + \cos \theta \cos \varphi \sin \psi)(-\sin \psi \sin \varphi + \cos \theta \cos \varphi \cos \psi)(-\sin \theta \cos \varphi) \\ (\sin \theta \sin \varphi) & (\sin \theta \cos \psi) & (\cos \theta) \end{bmatrix}$$

The students are urged to consult standard texts for further discussions on matrices and determinants of a more general nature.

1.7 Integral Theorems

A tensor-valued function $\mathfrak{J}(\underline{x})$ of position \underline{x} in a finite region R of the three-dimensional Euclidean space is said to be continuous at a point

\underline{x}^0 in R if the $\lim_{\alpha \rightarrow 0} \underline{T}(\underline{x}^0 + \alpha \underline{x})$ exists independently of the particular choice of the vector \underline{x} of a finite length, α being a real variable. Such a tensor-valued function defines a continuous tensor field in R if it is defined and continuous at every point in R . Let $\underline{T}(\underline{x}) = T_{ij}(\underline{x}) \underline{e}_i \underline{e}_j$ be a continuously differentiable tensor field of second order. The gradient of \underline{T} is defined as

$$\begin{aligned} \underline{\nabla} \underline{T} &= \underline{e}_i \frac{\partial}{\partial x_i} (T_{jk} \underline{e}_j \underline{e}_k) \\ &= \frac{\partial T_{jk}}{\partial x_i} \underline{e}_i \underline{e}_j \underline{e}_k \end{aligned} \quad (7.1)$$

and thus is a third order tensor-valued function in R . Equation (7.1) may be written as

$$\underline{\nabla} \underline{T} = T_{jk,i} \underline{e}_i \underline{e}_j \underline{e}_k \quad (7.1)$$

For a continuously differentiable, n th order tensor field $\underline{T}(\underline{x})$ in R , the quantity $\underline{\nabla} * \underline{T}(\underline{x})$ is a tensor of order $n - 1$, n , or $n + 1$ according to whether the symbol $*$ defines a dot, a cross, or a tensor product, that is, according to whether we have $\underline{\nabla} \cdot \underline{T}(\underline{x})$, $\underline{\nabla} \times \underline{T}(\underline{x})$, or $\underline{\nabla} \underline{T}(\underline{x})$. Let R be a convex region, bounded by a regular surface S which possesses a piecewise continuously turning tangent plane. The following integral theorem, called the Gauss theorem, then holds identically:

$$\int_R \underline{\nabla} * \underline{T} dV = \int_S \underline{n} * \underline{T} dS \quad , \quad (7.2a)$$

where dV is the elementary volume of R , dS is the elementary surface of S , and \underline{n} is the exterior unit normal to S . The symbol $*$ in (7.2a) may, of course, be interpreted as a dot, a cross, or a tensor product according to the considered case. The validity of (7.2a) may be proved as follows: The left side of (7.2a) may be written as

$$\begin{aligned} \int_R \underline{\nabla} * \underline{\mathcal{J}} dV &= \int_R \frac{\partial}{\partial x_i} (\underline{e}_i * \underline{\mathcal{J}}) dV = \\ &= \int_R \left[\frac{\partial}{\partial x_1} (\underline{e}_1 * \underline{\mathcal{J}}) + \frac{\partial}{\partial x_2} (\underline{e}_2 * \underline{\mathcal{J}}) + \right. \\ &\quad \left. \frac{\partial}{\partial x_3} (\underline{e}_3 * \underline{\mathcal{J}}) \right] dx_1 dx_2 dx_3 . \end{aligned} \tag{7.3}$$

Since the region R is assumed to be convex, every straight line parallel to the coordinate axes intersects the surface S in, at most, two points. Consider now the last term in the right side of equation (7.3), and noting that $(\underline{n} \cdot \underline{e}_3) dS^u = dx_1 dx_2$ on S^u and $(\underline{n} \cdot \underline{e}_3) dS^l = -dx_1 dx_2$ on S^l (see Fig. 7.1), obtain

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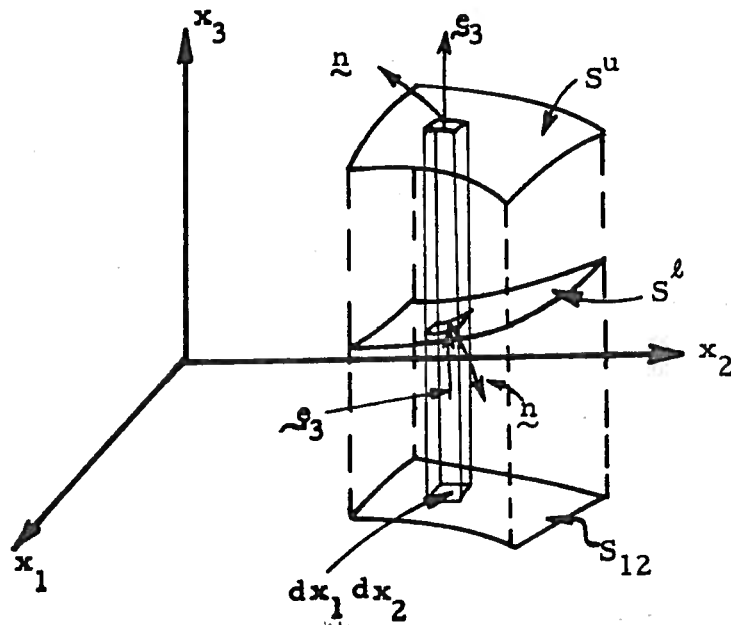


Figure 7.1

$$\begin{aligned}
 & \int_R \frac{\partial}{\partial x_3} (\underline{e}_3 * \underline{\mathcal{J}}) dx_1 dx_2 dx_3 \\
 &= \int_{S_{12}} dx_1 dx_2 \int_{x_3}^{x_3^u} \frac{\partial}{\partial x_3} (\underline{e}_3 * \underline{\mathcal{J}}) dx_3 \\
 &= \int_{S^u} (\underline{e}_3 * \underline{\mathcal{J}}) (\underline{n} \cdot \underline{e}_3) dS \\
 &+ \int_{S^l} (\underline{e}_3 * \underline{\mathcal{J}}) (\underline{n} \cdot \underline{e}_3) dS \\
 &= \int_S (\underline{e}_3 * \underline{\mathcal{J}}) (\underline{n} \cdot \underline{e}_3) dS .
 \end{aligned}
 \tag{7.4a}$$

Similarly, for the first and the second integrals in the right side of equation (7.3), we obtain

$$\int_R \frac{\partial}{\partial x_1} (\underline{e}_1 * \underline{\mathcal{J}}) dx_1 dx_2 dx_3 = \int_S (\underline{e}_1 * \underline{\mathcal{J}}) (\underline{n} \cdot \underline{e}_1) dS, \quad (7.4a)$$

and

$$\int_R \frac{\partial}{\partial x_2} (\underline{e}_2 * \underline{\mathcal{J}}) dx_1 dx_2 dx_3 = \int_S (\underline{e}_2 * \underline{\mathcal{J}}) (\underline{n} \cdot \underline{e}_2) dS. \quad (7.4b)$$

Noting that $\underline{n} = (\underline{n} \cdot \underline{e}_i) \underline{e}_i$, substitution from (7.4) into (7.3) yields

$$\begin{aligned} \int_R \underline{\nabla} * \underline{\mathcal{J}} dV &= \int_S \left[(\underline{n} \cdot \underline{e}_1) \underline{e}_1 * \underline{\mathcal{J}} + (\underline{n} \cdot \underline{e}_2) \underline{e}_2 * \underline{\mathcal{J}} + (\underline{n} \cdot \underline{e}_3) \underline{e}_3 * \underline{\mathcal{J}} \right] dS \\ &= \int_S \underline{n} * \underline{\mathcal{J}} dS \end{aligned} \quad (7.2b)$$

which proves the theorem of Gauss.

It should be noted that, although the Gauss theorem was proven for a convex region and a continuously differentiable tensor field $\underline{\mathcal{J}}(\underline{x})$, it is also valid when R can be decomposed into a finite number of such regions, or if R can be obtained as the limit of a sum of such parts. Moreover, if $\underline{\mathcal{J}}(\underline{x})$ is a piecewise differentiable tensor-valued function in R , the Gauss theorem can be written for each subregion in which $\underline{\mathcal{J}}$ is continuously differentiable. When adding, the surface integral over a common boundary of two such subregions vanishes only if $\underline{\mathcal{J}}$ is continuous across this surface, otherwise this contribution must also be accounted for in the final sum.

For a scalar field $\phi(\underline{x})$ and a vector field $\underline{v}(\underline{x})$, the Gauss theorem (7.2a) may be written in the following more familiar forms:

$$\int_R \underline{\nabla} \phi \, dV = \int_S \phi \, \underline{n} dS \quad (7.5a)$$

$$\int_R \underline{\nabla} \cdot \underline{v} \, dV = \int_S \underline{n} \cdot \underline{v} \, dS \quad (7.5b)$$

$$\int_R \underline{\nabla} \times \underline{v} \, dV = \int_S \underline{n} \times \underline{v} \, dS \quad (7.5c)$$

We now consider a tensor-valued function which is defined and continuously differentiable on an orientable, open surface S that is bounded by a closed curve C . With \underline{n} denoting the unit normal to S , we assign a positive sense to the curve C by means of its unit tangent vector \underline{t} that turns according to a right-handed screw which is pointing in the positive direction of \underline{n} . With these preliminaries, the theorem of Stokes can be written as

$$\int_S (\underline{n} \times \underline{\nabla}) * \underline{\mathcal{I}} \, dS = \oint_C \underline{t} * \underline{\mathcal{I}} \, dC, \quad (7.6)$$

where $\underline{n} \times \underline{\nabla} \equiv (n_i \underline{e}_i) \times (\underline{e}_j \frac{\partial}{\partial x_j}) = (e_{ijk} n_i \frac{\partial}{\partial x_j}) \underline{e}_k$; the symbol "*", as before, may stand for a dot, or a cross, or a tensor product; dC is the elementary length of the curve C . In (7.6), the line integral is to be taken in the positive direction of C .