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## Overall dynamic constitutive relations of layered elastic composites

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## ABSTRACT

A method for the homogenization of a layered elastic composite is presented. It allows direct, consistent, and accurate evaluation of the averaged overall frequency-dependent dynamic material constitutive relations without the need for a point-wise solution of the field equations. When the spatial variation of the field variables is restricted by Bloch-form (Floquet-form) periodicity, then these relations together with the overall conservation and kinematical equations accurately yield the displacement or stress mode-shapes and, necessarily, the dispersion relations. The method can also give the point-wise solution of the elastodynamic field equations (to any desired degree of accuracy), which, however, is not required for the calculation of the average overall properties. The resulting overall dynamic constitutive relations are general and need not be restricted by the Bloch-form periodicity.

The formulation is based on micromechanical modeling of a representative unit cell of the composite. For waves in periodic layered composites, the overall effective mass-density and compliance (stiffness) are always real-valued whether or not the corresponding unit cell (representative volume element used as a unit cell) is geometrically and/or materially symmetric. The average strain and linear momentum are coupled and the coupling constitutive parameters are always each others' complex conjugates. We separate the overall constitutive relations, which depend only on the composition and structure of the unit cell, from the overall field equations which hold for any elastic composite; i.e., we use only the *local* field equations and material properties to deduce the *overall* constitutive relations. Finally, we present solved numerical examples to further clarify the structure of the averaged constitutive relations and to bring out the correspondence of the current method with recently published results.

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## 1. Introduction

The recent interest in the characterization of the overall dynamic properties of composites with tailored microstructure necessitates a systematic homogenization procedure to express the dynamic response of an elastic composite in terms of its average effective compliance and density. The resulting homogenized parameters, once restricted by a Bloch-form periodicity, must give the composite's dispersion relations as well as the corresponding strain or stress mode-shapes.

The elastostatic response of composites has been long understood to be non-local in space (see Hill, 1965; Beran, 1968; Willis, 1977, 1983; Diener et al., 1984; Bakhvalov and Panasenko, 1989; Torquato and Haslach, 2002). But in the context of inhomogeneous elastodynamics the effective constitutive relations are non-local in both space and time (Willis, 1981a,b).

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Field integration-based homogenization for calculating these overall dynamic properties of composites has been proposed by a number of researchers. This approach requires a complete solution of the field equations; for electromagnetic waves see, e.g., Smith and Pendry (2006), Amirkhizi and Nemat-Nasser (2008), Bensoussan et al. (1978) and Sihvola (1999); and for elastodynamic waves see Willis (2009) who has presented a homogenization method based on an ensemble averaging technique of the ‘Bloch’ reduced form of the wave propagating in a periodic composite (see also Milton and Willis, 2007; Willis, 2011). An analogous approach for the calculation of effective compliance and mass-density of a layered composite was worked out in some detail by Nemat-Nasser et al. (2011) who showed that the corresponding results yield the exact dispersion relations.

In the present paper we propose a method for homogenization of layered elastic composites that does not require the point-wise solution of the field equations but directly provides the overall frequency-dependent dynamic material parameters. The method is inspired by the micromechanical homogenization of the static properties of a typical unit cell of a composite, originally proposed by Nemat-Nasser and Taya (1981), Nemat-Nasser et al. (1982), and Iwakuma and Nemat-Nasser (1983); see Nemat-Nasser and Hori (1999) for a comprehensive account. A similar homogenization method has been used by Amirkhizi and Nemat-Nasser (2008) to calculate the effective electromagnetic properties of a periodic composite. The current approach clearly distinguishes the overall constitutive relations, which should reflect the composition of the unit cell, from the overall conservation and kinematical relations which should hold for any elastic composite with any unit cell. In other words, while we do use the *local* (at each point within the unit cell) material properties and do enforce the conservation laws at the local level to obtain the *overall* constitutive relations, the *overall* conservation and kinematical relations are *not* used to derive these overall constitutive relations.

In what follows we outline our homogenization approach for a layered composite. We compare, by way of an example of a 2-layered composite, the corresponding dispersion results with those obtained using the exact solution (Rytov, 1956). Their close correspondence indicates that our method can be used to calculate the dispersion relation for more complex two- and three-dimensional cases where exact solutions are not available (as we shall report elsewhere.) Here we also use our approach to homogenize a 4-layered composite and compare the results with those obtained by the field integration of the stress and displacement mode-shapes obtained using the mixed variational formulation of Nemat-Nasser (1972) and Nemat-Nasser et al. (1975). It is shown that the current method gives homogenized results which converge to the field integration-based homogenization results that are based on either the exact solution or the mixed variational formulation. Thus, the current method can be used to evaluate the effective dynamic parameters of complex unit cells, without *a priori* requiring explicit point-wise values of the field variables, but instead producing these quantities if desired.

## 2. Micromechanical homogenization of layered composites

Here we present a homogenization method based on a micromechanical consideration of the volume averages of the field variables, viewed as measurable macroscopic physical quantities. We express the solution to the elastodynamic equations of motion as the sum of the volume average and a disturbance field due to the heterogeneous composition of the unit cell:

$$Q = Q^0 + Q^d \quad (1)$$

where  $Q$  represents any of the field variables, stress,  $\sigma$ , or velocity,  $\dot{u}$ . The aim is to derive a set of constitutive relations for the overall averaged parts of the field variables, using the local elastodynamic equations of motion and constitutive relations. This then provides the homogenized frequency-dependent material parameters.

Consider harmonic waves in an unbounded elastic composite consisting of a collection of bonded, identical unit cells,  $\Omega = \{x : -a/2 \leq x < a/2\}$ , which are repeated in the  $x$ -direction, and hence constitute a periodic structure. In view of the periodicity of the composite, we have  $\rho(x) = \rho(x + m'a)$  and  $C(x) = C(x + m'a)$ ; here  $m'$  is an integer,  $\rho(x)$  is the density and  $C(x)$  is the modulus of elasticity. For time harmonic waves with frequency  $\omega$ , the field quantities are proportional to  $e^{\pm i\omega t}$ . For waves with arbitrary wavenumber  $q$  and unrelated frequency  $\omega$ , we express the field variables in the following form:

$$\hat{F}(x, t) = \text{Re}[F(x) \exp[i(qx - \omega t)]] \quad (2)$$

where  $\hat{F}$  represents the field variables, stress,  $\hat{\sigma}$ , strain,  $\hat{\varepsilon}$ , momentum,  $\hat{p}$ , displacement,  $\hat{u}$ , or velocity,  $\hat{\dot{u}}$ , whereas  $F$  represents their periodic parts ( $\sigma, \varepsilon, p, u, \dot{u}$ ). The representation equation (2), separates the macroscopic time harmonic from the microscopic part of the field variables. Even for a finite unit cell, the Fourier series solution of the microscopic part is periodic so that this solution satisfies,  $F(x) = F(x + m'a)$ . We emphasize here that the frequency and the wavenumber,  $\omega$  and  $q$ , are, at this point, unrelated and arbitrary.

The local conservation and kinematic relations are

$$\begin{aligned} \nabla \sigma &= -i\omega p \\ \nabla \dot{u} &= -i\omega \varepsilon \end{aligned} \quad (3)$$

where  $\nabla \rightarrow \partial/\partial x + iq$ . The corresponding constitutive relations are

$$\varepsilon = D(x)\sigma$$

$$p = \rho(x)\dot{u} \quad (4)$$

where  $D(x)$  is the compliance, the inverse of the elastic modulus, and  $\rho(x)$  is the mass-density. These local material parameters represent the structure and composition of the unit cell.

Now we replace the heterogeneous unit cell with a homogeneous one having uniform density  $\rho_0$  and compliance  $D_0$ . In order to reproduce the strain and momentum of the actual unit cell, field variables eigenstress  $\Sigma(x)$ , and eigenvelocity  $\dot{U}(x)$ , are introduced. These quantities are then calculated using the basic local field equations and constitutive relations. The idea stems from the polarization stress or strain that was originally proposed by Hashin (1959) and further developed by Hashin and Shtrikman (1962a,b) and later by others, in order to construct energy-based bounds for the composite's overall elastic moduli. The basic tool in these works has been the result obtained by Eshelby (1957) in three dimensions, and earlier by Hardiman (1954) in two dimensions, that the stress and strain are constant within an ellipsoidal (elliptical in two dimensions) region of an infinitely extended uniform elastic medium when that region undergoes a uniform transformation corresponding to a uniform inelastic strain.

In this paper, we present a method that can be used to directly calculate the homogenized properties without needing the point-wise values of the field quantities, yet the method can actually provide these point-wise field values (to any degree of accuracy) if desired. For this, we require that the actual values of the field variables at every point within the homogenized and the original heterogeneous unit cell be exactly the same. To ensure this, we require that the following consistency conditions hold at every point within the unit cell:

$$\varepsilon(x) = D(x)\sigma(x) = D_0(\sigma(x) - \Sigma(x))$$

$$p(x) = \rho(x)\dot{u}(x) = \rho_0(\dot{u}(x) - \dot{U}(x)) \quad (5)$$

The eigenstress and eigenvelocity fields  $\Sigma(x)$  and  $\dot{U}(x)$  are zero in regions where the material properties of the heterogeneous unit cell are equal to the uniform material properties,  $D_0$  and  $\rho_0$  (which can have any positive values without affecting the final results).

Averaging the consistency conditions, Eqs. (5), we obtain

$$\langle \varepsilon \rangle = D_0[\langle \sigma \rangle - \langle \Sigma \rangle] \quad (6)$$

$$\langle p \rangle = \rho_0[\langle \dot{u} \rangle - \langle \dot{U} \rangle] \quad (7)$$

In this approach, the calculation of the effective properties does not require point-wise values of the eigenstress and eigenvelocity fields, only the volume averages  $\langle \Sigma \rangle$  and  $\langle \dot{U} \rangle$ . To calculate these, we first formulate a pair of integral equations for these eigenfields, and then, instead of seeking to find the point-wise solution of the field variables, we calculate the volume averages of the eigenfields in terms of the average stress and velocity,  $\langle \sigma \rangle$  and  $\langle \dot{u} \rangle$ .

From Eqs. (3) and (5) we have

$$\nabla^2 \sigma + v^2 \sigma = v^2 \Sigma - \frac{v^2}{i\omega D_0} \nabla \dot{U} \quad (8)$$

$$\nabla^2 \dot{u} + v^2 \dot{u} = v^2 \dot{U} - \frac{v^2}{i\omega \rho_0} \nabla \Sigma \quad (9)$$

where  $v^2 = \omega^2 \rho_0 D_0$ . We now consider a Fourier series solution of the above equations. With  $\xi = \pm 2n\pi/a, n \neq 0$ , we set

$$F(x) = F^0 + F^p = \langle F \rangle_\Omega + \sum_{\xi \neq 0} F(\xi) e^{i\xi x} \quad (10)$$

$$\langle F \rangle_\Omega = \frac{1}{\Omega} \int_\Omega F(x) dx \quad (11)$$

$$F(\xi) = \frac{1}{\Omega} \int_\Omega F(x) e^{-i\xi x} dx \quad (12)$$

where  $\langle F \rangle_\Omega$  represents the averaged value of the field variable,  $F$ , over the unit cell, and  $F(\xi)$  represents the Fourier coefficient of the corresponding local disturbance due to heterogeneity. From Eqs. (8) and (9) we obtain the following Fourier coefficients for the stress and velocity fields:

$$\sigma(\xi) = \frac{v^2}{[v^2 - (\xi + q)^2]} \Sigma(\xi) - \frac{v^2(\xi + q)}{\omega D_0 [v^2 - (\xi + q)^2]} \dot{U}(\xi) \quad (13)$$

$$\dot{u}(\xi) = \frac{v^2}{[v^2 - (\xi + q)^2]} \dot{U}(\xi) - \frac{v^2(\xi + q)}{\omega \rho_0 [v^2 - (\xi + q)^2]} \Sigma(\xi) \quad (14)$$

Therefore, the stress and velocity fields can be expressed as a sum of their average and their periodic components

$$\sigma(x) = \langle \sigma \rangle + \sum_{\xi \neq 0} e^{i\xi x} \left[ A(\xi) \frac{1}{\Omega} \int_{\Omega} \Sigma(y) e^{-i\xi y} dy - \frac{B(\xi)}{\omega D_0} \frac{1}{\Omega} \int_{\Omega} \dot{U}(y) e^{-i\xi y} dy \right] \tag{15}$$

$$\dot{u}(x) = \langle \dot{u} \rangle + \sum_{\xi \neq 0} e^{i\xi x} \left[ A(\xi) \frac{1}{\Omega} \int_{\Omega} \dot{U}(y) e^{-i\xi y} dy - \frac{B(\xi)}{\omega \rho_0} \frac{1}{\Omega} \int_{\Omega} \Sigma(y) e^{-i\xi y} dy \right] \tag{16}$$

$$A(\xi) = \frac{v^2}{[v^2 - (\xi + q)^2]}, \quad B(\xi) = \frac{v^2(\xi + q)}{[v^2 - (\xi + q)^2]} \tag{17}$$

where  $\langle \dot{u} \rangle$  and  $\langle \sigma \rangle$  are the average values of the velocity and stress fields, respectively, taken over a unit cell. To make the homogenized unit cell point-wise equivalent to the original heterogeneous unit cell, we can combine the above equations with Eqs. (5) to obtain a set of two coupled integral equations for the eigenfields,  $\Sigma(x)$  and  $\dot{U}(x)$ , which homogenize the unit cell point-wise, yielding the exact solution of the field quantities. Our aim, however, is *not* the point-wise representation of the heterogeneous medium, but, rather, it is the determination of the averaged field values.

To calculate  $\langle \Sigma \rangle$  and  $\langle \dot{U} \rangle$  we divide the unit cell into  $\bar{\alpha}$  subregions  $\Omega_{\alpha}$ . Then, we average the periodic fields over each such subregion to obtain

$$\langle \sigma^p \rangle_{\Omega_{\alpha}} = \sigma_{\alpha}^p = \frac{1}{\Omega_{\alpha}} \int_{\Omega_{\alpha}} \sigma^p(x) dx = \sum_{\xi \neq 0} g_{\alpha}(\xi) \left( A(\xi) \frac{1}{\Omega} \int_{\Omega} \Sigma(y) e^{-i\xi y} dy - \frac{B(\xi)}{\omega D_0} \frac{1}{\Omega} \int_{\Omega} \dot{U}(y) e^{-i\xi y} dy \right) \tag{18}$$

$$\langle \dot{u}^p \rangle_{\Omega_{\alpha}} = \dot{u}_{\alpha}^p = \frac{1}{\Omega_{\alpha}} \int_{\Omega_{\alpha}} \dot{u}^p(x) dx = \sum_{\xi \neq 0} g_{\alpha}(\xi) \left( A(\xi) \frac{1}{\Omega} \int_{\Omega} \dot{U}(y) e^{-i\xi y} dy - \frac{B(\xi)}{\omega \rho_0} \frac{1}{\Omega} \int_{\Omega} \Sigma(y) e^{-i\xi y} dy \right) \tag{19}$$

$$g_{\alpha}(\xi) = \frac{1}{\Omega_{\alpha}} \int_{\Omega_{\alpha}} e^{i\xi x} dx \tag{20}$$

We now replace the integrals in Eqs. (18) and (19) by their equivalent finite sums and set

$$\frac{1}{\Omega} \int_{\Omega} F(y) e^{-i\xi y} dy \approx \sum_{\beta} f_{\beta} g_{\beta}(-\xi) F_{\beta}$$

$$f_{\beta} = \frac{\Omega_{\beta}}{\Omega}$$

$$F_{\beta} = \langle F \rangle_{\Omega_{\beta}} \tag{21}$$

Eqs. (18) and (19) then yield the following expressions:

$$\sigma_{\alpha}^p = A_{\alpha\beta} \Sigma_{\beta} - \frac{1}{\omega D_0} B_{\alpha\beta} \dot{U}_{\beta} \tag{22}$$

$$\dot{u}_{\alpha}^p = A_{\alpha\beta} \dot{U}_{\beta} - \frac{1}{\omega \rho_0} B_{\alpha\beta} \Sigma_{\beta} \tag{23}$$

where the repeated index,  $\beta$ , is summed over the number of subregions,  $\beta = 1, \dots, \bar{\alpha}$ . The coefficient matrices in the above equations are defined by

$$A_{\alpha\beta} = \sum_{\xi \neq 0} g_{\alpha}(\xi) f_{\beta} g_{\beta}(-\xi) A(\xi)$$

$$B_{\alpha\beta} = \sum_{\xi \neq 0} g_{\alpha}(\xi) f_{\beta} g_{\beta}(-\xi) B(\xi) \tag{24}$$

In these equations,  $\beta$  is not summed. Averaging the consistency conditions over each subregion  $\alpha$  and using Eqs. (22) and (23), we have

$$f_{\alpha} \langle \sigma \rangle = - \left[ \bar{A}_{\alpha\beta} + \frac{f_{\alpha} D_0}{D_{\alpha} - D_0} \delta_{\alpha\beta} \right] \Sigma_{\beta} + \frac{1}{\omega D_0} \bar{B}_{\alpha\beta} \dot{U}_{\beta} \tag{25}$$

$$f_{\alpha} \langle \dot{u} \rangle = \frac{1}{\omega \rho_0} \bar{B}_{\alpha\beta} \Sigma_{\beta} - \left[ \bar{A}_{\alpha\beta} + \frac{f_{\alpha} \rho_0}{\rho_{\alpha} - \rho_0} \delta_{\alpha\beta} \right] \dot{U}_{\beta} \tag{26}$$

$$\bar{A}_{\alpha\beta} = f_{\alpha} A_{\alpha\beta}$$

$$\bar{B}_{\alpha\beta} = f_{\alpha} B_{\alpha\beta} \quad \alpha \text{ not summed} \tag{27}$$

Note that  $\bar{A}_{\alpha\beta}$  and  $\bar{B}_{\alpha\beta}$  are geometric quantities, independent of the material properties of the unit cell. Expressions (25, 26) are  $2\bar{x}$  linear equations which can be solved for  $\Sigma_\beta$  and  $\dot{U}_\beta$  in terms of the average stress  $\langle \sigma \rangle$  and average velocity  $\langle \dot{u} \rangle$ . We express the solution in the following form:

$$\{\Sigma\} = \{\Phi\}\langle \sigma \rangle + \frac{1}{D_0}\{\Psi\}\langle \dot{u} \rangle$$

$$\{\dot{U}\} = \frac{1}{\rho_0}\{\Theta\}\langle \sigma \rangle + \{\Gamma\}\langle \dot{u} \rangle \quad (28)$$

where explicit expressions for  $\{\Phi\}$ ,  $\{\Psi\}$ ,  $\{\Theta\}$ , and  $\{\Gamma\}$  are provided in Appendix B. Now the piecewise constant eigenfields, given in the above equations, can be averaged over the unit cell to express  $\langle \Sigma \rangle$  and  $\langle \dot{U} \rangle$  in terms of  $\langle \sigma \rangle$  and  $\langle \dot{u} \rangle$ .

The averaged consistency conditions, Eqs. (6) and (7), can then be expressed as

$$\langle \varepsilon \rangle = \bar{D}\langle \sigma \rangle + \bar{S}_1\langle \dot{u} \rangle \quad (29)$$

$$\langle p \rangle = \bar{S}_2\langle \sigma \rangle + \bar{\rho}\langle \dot{u} \rangle \quad (30)$$

Eqs. (29) and (30) are our final constitutive relations for the homogenized composite. In these equations,  $\bar{D}$  and  $\bar{\rho}$  are the overall homogenized composite's compliance and mass-density, respectively. As is shown in Appendix B,  $\bar{D}$  and  $\bar{\rho}$  are always real-valued for a layered composite. The coupling terms,  $\bar{S}_1$  and  $\bar{S}_2$ , are always each others' complex conjugate. These attributes hold whether the unit cell of the layered composite is symmetric or non-symmetric. The coupling terms of course vanish if the unit cell is homogeneous. They are real-valued and equal when the unit cell has reflective symmetry. These relations also ensure that the total energy given by

$$\begin{aligned} E &= \frac{1}{2}[\langle \sigma \rangle^* \langle \varepsilon \rangle + \langle \dot{u} \rangle^* \langle p \rangle] \\ &= \frac{1}{2}[\bar{D}\langle \sigma \rangle \langle \sigma \rangle^* + \bar{\rho}\langle \dot{u} \rangle \langle \dot{u} \rangle^* + \bar{S}_1\langle \sigma \rangle^* \langle \dot{u} \rangle + \bar{S}_2\langle \sigma \rangle \langle \dot{u} \rangle^*] \\ &= \frac{1}{2}[\bar{D}\langle \sigma \rangle \langle \sigma \rangle^* + \bar{\rho}\langle \dot{u} \rangle \langle \dot{u} \rangle^* + \bar{S}_1\langle \sigma \rangle^* \langle \dot{u} \rangle + (\bar{S}_1\langle \sigma \rangle^* \langle \dot{u} \rangle)^*] \end{aligned} \quad (31)$$

is always real-valued.

### 2.1. Bloch-form overall spatial variation and dispersion relations

The wavenumber  $q$  and frequency  $\omega$  have been assumed to be independent till now. We now consider the special case of an infinite homogenized elastic solid with a layered microstructure, and seek conditions under which it supports periodic waves of the Bloch-form spatial variation of the following form:

$$\begin{aligned} \langle \sigma \rangle(x) &= \langle \sigma \rangle e^{iqx}, & \langle \dot{u} \rangle(x) &= \langle \dot{u} \rangle e^{iqx} \\ \langle \varepsilon \rangle(x) &= \langle \varepsilon \rangle e^{iqx}, & \langle p \rangle(x) &= \langle p \rangle e^{iqx} \end{aligned} \quad (32)$$

The overall field equations then become

$$\frac{d}{dx}(\langle \sigma \rangle e^{iqx}) = -i\omega \langle p \rangle e^{iqx}, \quad \frac{d}{dx}(\langle \dot{u} \rangle e^{iqx}) = -i\omega \langle \varepsilon \rangle e^{iqx} \quad (33)$$

which yield

$$q\langle \sigma \rangle = -\omega \langle p \rangle, \quad q\langle \dot{u} \rangle = -\omega \langle \varepsilon \rangle \quad (34)$$

These equations are combined with the constitutive relations (29) and (30) to eliminate three out of the four field variables and arrive at a single equation of the following form:

$$\hat{R}(q, \omega)\dot{u} = 0 \quad (35)$$

For nontrivial solutions to the above,  $\hat{R}(q, \omega) = 0$  which produces the dispersion relation of the composite. It can be shown that  $\hat{R}(q, \omega) = 0$  produces the following relation:

$$\left(\frac{\omega}{q}\right)^2 = v_p^2 = \frac{(1 + \nu_p \bar{S}_1)(1 + \nu_p \bar{S}_2)}{\bar{D}\bar{\rho}} \quad (36)$$

where  $\nu_p$  is the phase velocity. The above equation is used to evaluate the dispersion relation for a layered composite by substituting from Eq. (B.3). The real frequency–wavenumber pairs are then used to evaluate the effective parameters of Eqs. (29) and (30) and these parameters would have the properties discussed in Appendix B.

If we wish, as in Nemat-Nasser et al. (2011), to directly relate the overall strain,  $\langle \varepsilon \rangle$ , to the overall stress,  $\langle \sigma \rangle$ , and the overall momentum,  $\langle p \rangle$ , to the overall velocity,  $\langle \dot{u} \rangle$ , according to

$$\begin{aligned} \langle \varepsilon \rangle &= D^{\text{eff}} \langle \sigma \rangle \\ \langle p \rangle &= \rho^{\text{eff}} \langle \dot{u} \rangle \end{aligned} \tag{37}$$

then we have

$$D^{\text{eff}} = \frac{\bar{D}}{1 + \nu_p \bar{S}_1} \tag{38}$$

$$\rho^{\text{eff}} = \frac{\bar{\rho}}{1 + \nu_p \bar{S}_2} \tag{39}$$

and they do satisfy the same dispersion relation, Eq. (36), now rewritten as

$$\left(\frac{\omega}{q}\right)^2 = \nu_p^2 = \frac{1}{D^{\text{eff}} \rho^{\text{eff}}} \tag{40}$$

In this representation however, the effective parameters defined by (37) are real only for symmetric (elastic) unit cells and become complex-valued for asymmetric unit cells. Also the parameters  $D^{\text{eff}}$  and  $\rho^{\text{eff}}$  are obtained using the special form of the field equations (34) while the constitutive relations (29) and (30) have broader applicability.

A significant advantage of our homogenization technique is that the point-wise solution of the field equations is not required *a priori*, and that, this solution can be obtained as a by-product of the approach. In fact, within each subregion, the value of each field variable is given by adding to each periodic part in Eqs. (22) and (23), the corresponding average value. We do not need explicit expressions for the mode-shapes, but rather can extract these from the final results, if desired. The dispersion relations are obtained by substituting Eqs. (29) and (30) into Eq. (36). These relations then relate the frequency,  $\omega$ , and the wavenumber,  $q$ , for the special Bloch-form of the periodic waves, as exemplified in what follows.

2.2. Example: dispersion calculation for a layered composite

To illustrate dispersion calculations by our micromechanical formulation, consider the case of a layered composite (Fig. 1) with harmonic waves traveling perpendicular to the layers.

The exact dispersion relation for longitudinal waves in a layered composite was given by Rytov (1956). Here we compare the dispersion curves calculated using our micromechanical formulation with those of the exact solution. A unit cell, for this case, consists of 1 layer of Material 1 and 1 layer of Material 2. We divide the unit cell into  $N=30$  subregions (15 divisions per layer) and use constant values for the eigenstress and eigenvelocity within each subregion.

The first three modes in the dispersion curve are compared in Fig. 2. It can be seen that the micromechanical formulation gives very acceptable results for the first three modes when appropriate discretization is used. There are indications that the results of the micromechanical formulation start diverging from the exact results as one considers higher frequencies. This is evidenced by the increasing difference in the results as one moves up the third branch. This is of

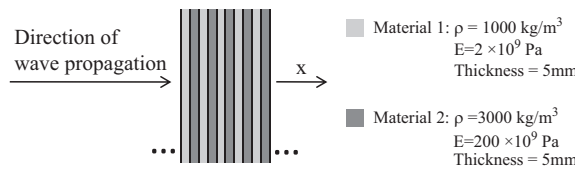


Fig. 1. Schematic of a bi-layered composite.

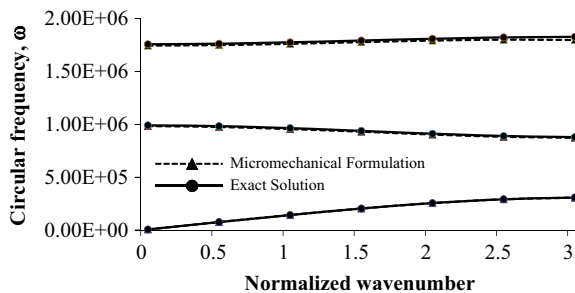


Fig. 2. Dispersion curve comparison of the micromechanical formulation with the exact results.

course an issue of discretization and the micromechanical formulation is expected to give accurate dispersion results at any frequency given a fine enough discretization.

It should be mentioned here that exact dispersion relations are only available for fairly simple geometries like layered composites. The lack of exact solutions for more complex cases, therefore, necessitates accurate and efficient numerical schemes. In a recent paper (Nemat-Nasser et al., 2011), a mixed method is used to obtain the point-wise values of the field variables and to formulate the corresponding integration-based homogenization of layered composites. This method was shown to be in good agreement with homogenization based upon the integration of field variables as calculated from the exact solution. Although homogenization based on the mixed variational method provides an accurate and efficient numerical alternative to using the exact solution, the field variables still need to be evaluated at multiple points within the unit cell. Here we show that the micromechanical approach to homogenization, in addition to precluding the need for point-wise evaluation of the field variables, also converges to homogenization results based on both the mixed variational method and the exact solution; a brief summary of integration of field variable based homogenization is presented in Appendix A.

### 3. Constitutive relations of layered composites

To illustrate the microstructural method of calculating the constitutive parameters of homogenized composites, examples of 4-layered symmetric and asymmetric composites are considered. These examples also serve to show that homogenization based on the micromechanical formulation converges to the results of homogenization based on the integration of field variables using the exact solution and/or using the results of the mixed variational method.

#### 3.1. The 4-layered symmetric composite: comparison with homogenization based on the mixed variational formulation

We consider a 4-layered composite in order to show that the micromechanical homogenization gives results which converge to homogenization based on the integration of field variables as derived from the mixed variational formulation. We stress that for this integration-based homogenization, the effective parameters are calculated from Eqs. (A.10) and (A.11) using the mode-shapes calculated from the approximate mixed variational formulation of Nemat-Nasser (1972), whereas for the micromechanical homogenization the constitutive parameters are calculated from Eqs. (38) and (39) directly. It should be noted that while the effective parameters calculated from Eqs. (A.10) and (A.11) require integration of the field variables through the unit cell, effective parameters calculated from Eqs. (38) and (39) do not require any integration. For symmetric unit cells such as Fig. 3a, Eqs. (38) and (39) result in real-valued effective parameters and produce the same constitutive relations (A.10) and (A.11) reported in the literature; Nemat-Nasser et al. (2011). Expressions (A.10) and (A.11) however do not clearly display the coupling between average strain and average velocity, and average momentum and average stress, which is non-zero even in the symmetric case (B.3). The micromechanical approach directly produces the effective parameters of Eq. (B.3) without any *a priori* assumption about the structure of the constitutive relations. These effective parameters can always be used to calculate  $D^{\text{eff}}$  and  $\rho^{\text{eff}}$ , if desired.

Fig. 3a shows one unit cell of the 4-layered composite under consideration. It is composed of a heavy and stiff layer sandwiched between two layers of a light and compliant material with the whole assembly being embedded in a heavy and stiff matrix. Here we calculate the effective dynamic properties of the composite using the micromechanical

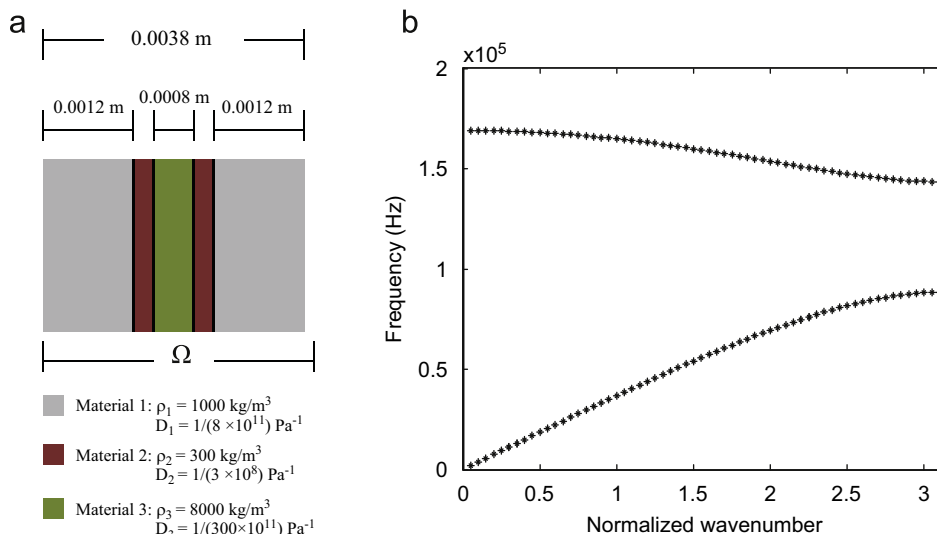
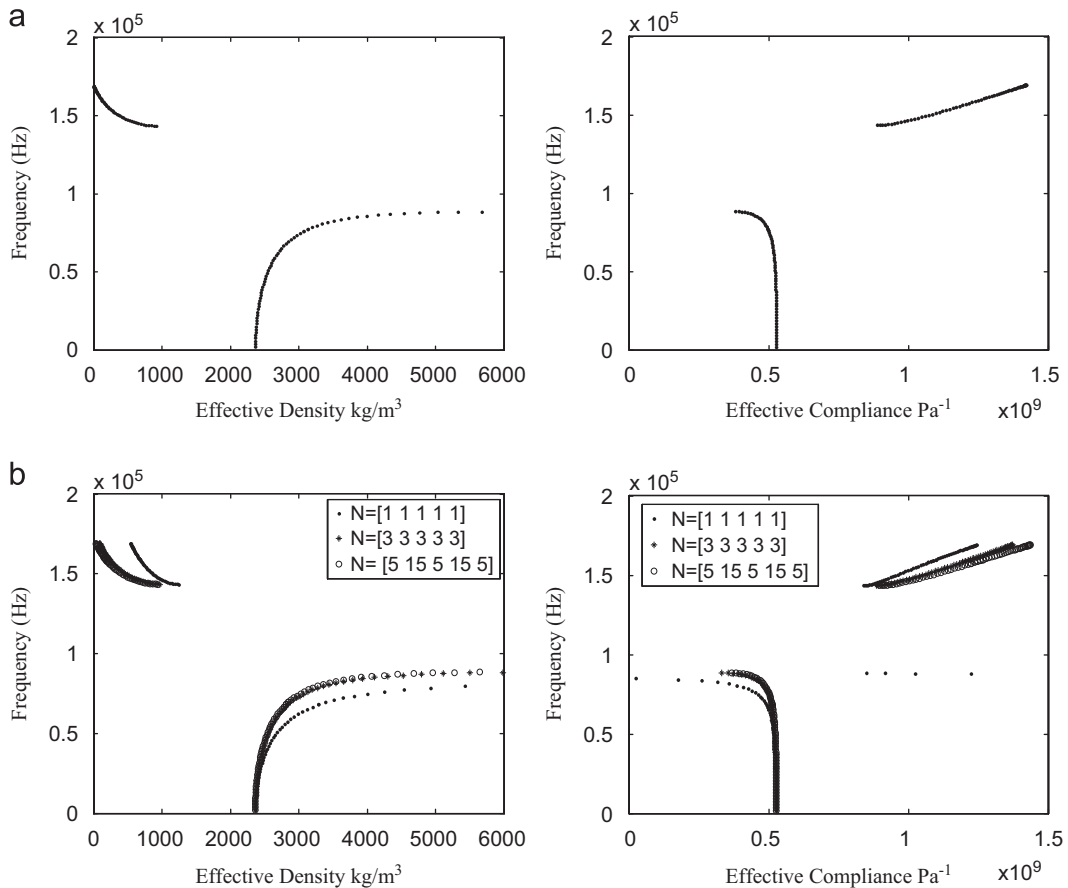


Fig. 3. The 4-layered symmetric composite. (a) Schematic of a unit cell. (b) Dispersion curve for the first two propagating branches.



**Fig. 4.** Effective parameters comparison. (a) Homogenization results from field integral method as applied to the mixed variational formulation. (b) Homogenization results from the micromechanical method with increasingly more refined discretization.

method (38) and (39) with increasingly more refined discretization and compare the corresponding homogenization results with those obtained from Eqs. (A.10) and (A.11).

Fig. 4a shows the homogenization results for the first two propagating branches calculated from the integration of field variables as derived from the mixed variational formulation. Fig. 4b shows the homogenization results from the micromechanical formulation. This example also serves to show that the initial choice of the homogenizing parameters ( $\rho_0, D_0$ ) is immaterial to the final homogenization results. In this example  $\rho_0$  and  $D_0$  are taken to be the average properties of layers 1 and 2. Therefore, the discretization of heterogeneity is applied to the entire unit cell and every layer is discretized. The effect of increasingly more refined discretization is shown in Fig. 4b. The number of terms in the Fourier expansion ( $\xi$ ) is kept constant at 10 for each level of discretization. From Fig. 4b, it can be seen that when each layer is considered as just one subregion of constant eigenstress and eigenvelocity ( $N=[1,1,1,1]$ ), the homogenization results deviate significantly from the results of the field integration method. As finer discretization is introduced in each layer ( $N=[3,3,3,3]$ ), homogenization based on the micromechanical method tends to converge to the results of the integration method using the mixed variational formulation. For a suitably fine discretization, especially for the case when the more compliant layer is modeled with greater number of subregions ( $N=[5,15,5,15]$ ), the results of the micromechanical method approach those of the mixed formulation, with any desired accuracy.

### 3.2. The 4-layered composite with an asymmetric unit cell

We now present an asymmetric unit cell which serves to illustrate the nature of the coupling parameters in Eq. (B.3). For a symmetric unit cell, all effective parameters calculated from Eq. (B.3) are real-valued and yield real-valued  $D^{\text{eff}}$  and  $\rho^{\text{eff}}$ . It is for the asymmetric unit cell that the coupling terms in Eq. (B.3) become complex-valued resulting in complex-valued  $D^{\text{eff}}$  and  $\rho^{\text{eff}}$ .

Fig. 5a shows one unit cell of the 4-layered asymmetric composite under consideration. It is composed of a heavy and stiff layer sandwiched between two equally thick layers made of different compliant materials, with this assembly being



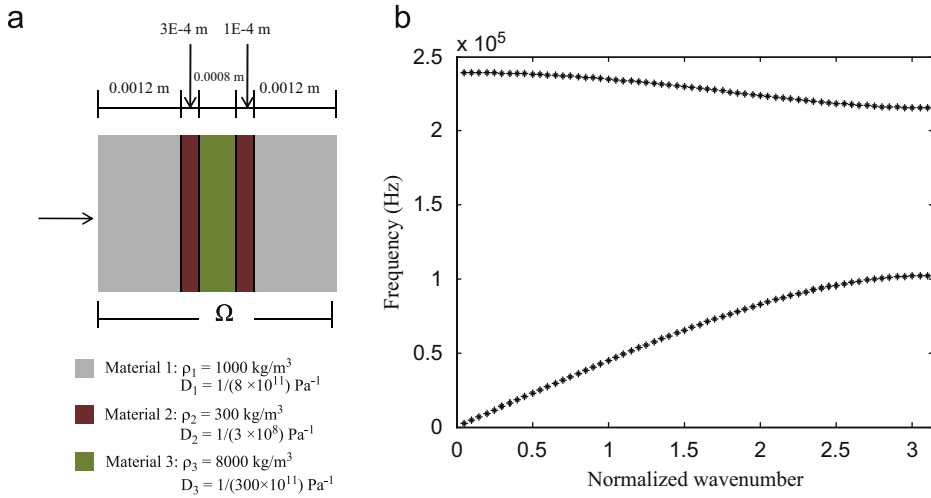


Fig. 5. The 4-layered asymmetric composite. (a) Schematic of a unit cell. (b) Dispersion curve for the first two propagating branches.

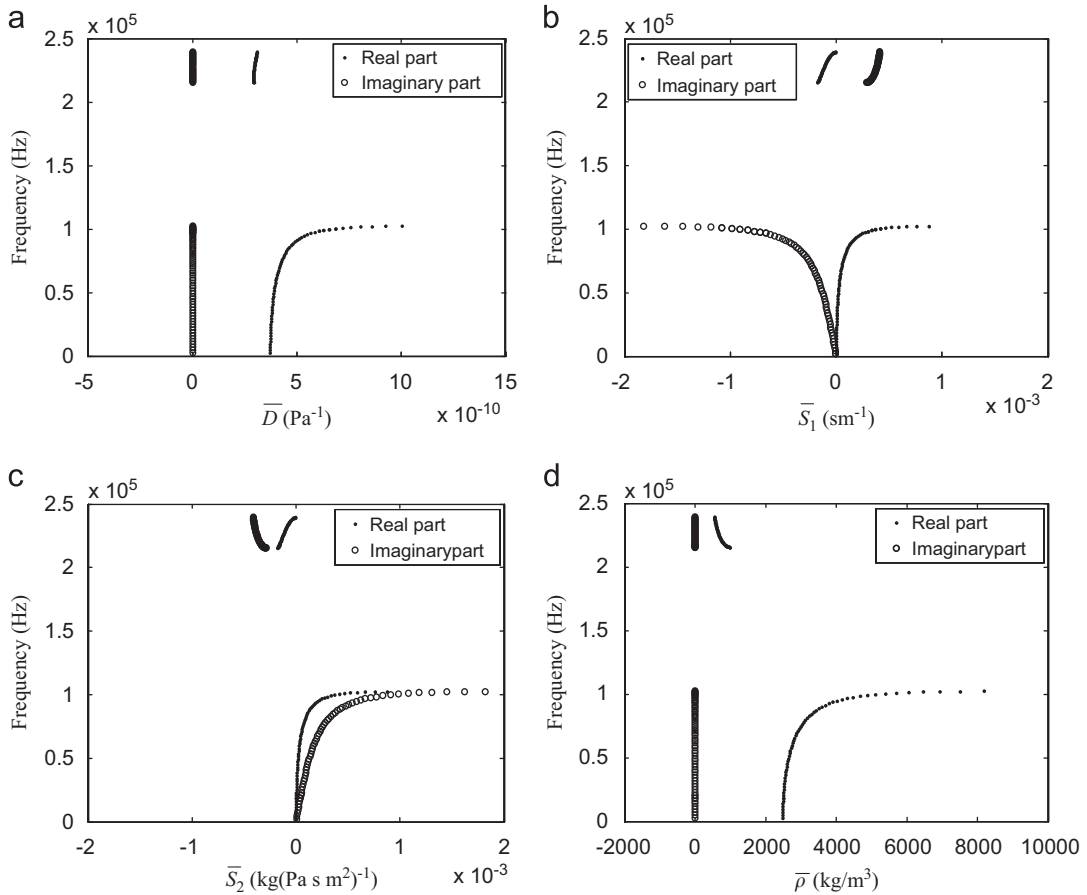


Fig. 6. Effective parameters calculated from the micromechanical formulation. (a)  $\bar{D}$ . (b)  $\bar{S}_1$ . (c)  $\bar{S}_2$ . (d)  $\bar{\rho}$ .

embedded in a heavy and stiff matrix. Here we calculate the effective dynamic properties of the composite using the micromechanical method ((38) and (39)).

Fig. 6 shows the homogenization results for the first two propagating branches calculated from the micromechanical formulation (Eq. (B.3)). In this example  $\rho_0$  and  $D_0$  are taken to be the properties of layer 1. Therefore, the discretization of heterogeneity applies to the central three layers. The number of terms in the Fourier expansion ( $\xi$ ) is kept constant at 10

and the discretization level of the central three layers is ( $N=[15,10,15]$ ). The real and imaginary parts of the calculated parameters are indicated with different symbols. Fig. 6a,d clearly shows that  $\bar{D}$  and  $\bar{p}$  are purely real for all propagating frequencies and Fig. 6b,c shows that  $\bar{S}_1$  and  $\bar{S}_2$  are complex conjugates of each other.

#### 4. Conclusions

A homogenization method based on micromechanical considerations is presented. This method does not require the point-wise values of the elastodynamic field variables within a unit cell, instead, it can produce those values to any desired degree of accuracy, if needed. Using the *local* conservation and kinematical relations together with the *local* constitutive relations, we have systematically deduced the overall constitutive relations for the homogenized elastic solid that depend on the frequency and wavenumber as two independent parameters. These parameters can be related to one another in solving any special overall initial-boundary-value problems associated with the homogenized continuum. For example, for a Bloch-form wave propagating in an infinite homogenized continuum (microstructurally layered periodic composite), one arrives at the dispersion relations that connect the wavenumber to the wave frequency, as necessary conditions for the existence of such solutions. In general, for elastic waves in a layered composite, the average strain and average momentum are given in terms of the average stress and average velocity by a pair of coupled constitutive relations. We have proved that the overall compliance and mass-density in these relations are always real-valued for a layered composite, and that the corresponding coupling terms are always complex conjugates of each other. For a symmetric unit cell of a layered composite, the coupling terms are real-valued and equal to one another.

For illustration, we have used the overall constitutive relations to calculate the dispersion relation of a layered composite and have shown that the results converge to the exact solution for the case of a 2-layered composite. For the special problem of a Bloch-form wave propagating in an infinite homogenized solid, one may use the overall conservation and kinematical relations to define another set of effective parameters,  $D^{\text{eff}}$  and  $\rho^{\text{eff}}$ , that directly relate the average strain to the average stress, and the average momentum to the average velocity, similarly to the corresponding quasi-static case. We have shown that these parameters are complex-valued for any non-symmetric unit cell. We have presented a symmetric 4-layered example showing that these parameters as calculated from the micromechanical method converge to those calculated by the integration of field variables approach of Nemat-Nasser et al. (2011). This also means that the results presented in this paper are consistent with the results of the ensemble averaging technique of Willis (2009) although an explicit proof for the same is not presented here. The micromechanical method we presented here may be used to homogenize unit cells of more complex microstructures in two- and three-dimensional composites where the exact dispersion relations are not known and the point-wise solution to the field equations is not available (as we shall show in a subsequent work). Finally we presented an asymmetric 4-layered example where we have shown that the coupling terms are complex conjugates of each other and  $\bar{D}$  and  $\bar{p}$  are real-valued, whereas the corresponding  $D^{\text{eff}}$  and  $\rho^{\text{eff}}$  are complex-valued.

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#### Appendix A. Homogenization by integration of field variables

Nemat-Nasser et al. (2011) proposed a homogenization method of periodic elastic composites based upon the integration of field variables. Here we present a brief summary of the basic equations. For harmonic waves traveling in a layered composite with a periodic unit cell  $\Omega = \{x : -a/2 \leq x < a/2\}$  the field variables (displacement, velocity, stress, and momentum) take the Bloch form given by Eq. (2). The dynamic equilibrium gives

$$\nabla \hat{\sigma} + i\omega \hat{p} = 0 \quad (\text{A.1})$$

where  $\nabla$  denotes differentiation with respect to  $x$ .

##### A.1. Effective properties

Multiply Eq. (A.1) by  $e^{-iqX}$  and use Eq. (2) to obtain

$$\nabla(\sigma(x)e^{iq(x-X)}) + i\omega p(x)e^{iq(x-X)} = 0 \quad (\text{A.2})$$

Introduce the change of variable  $y=x-X$  and average with respect to  $X$  over a unit cell to obtain

$$\nabla_y(\langle \sigma \rangle e^{iqy}) + i\omega \langle p \rangle e^{iqy} = 0 \quad (\text{A.3})$$

where

$$\langle \sigma \rangle = \frac{1}{a} \int_{-a/2}^{+a/2} \sigma(x) dx, \quad \langle p \rangle = \frac{1}{a} \int_{-a/2}^{+a/2} p(x) dx \quad (\text{A.4})$$

Now define the *mean stress* and *mean momentum density* as

$$\langle \hat{\sigma} \rangle(x) = \langle \sigma \rangle e^{iqx}; \quad \langle \hat{p} \rangle(x) = \langle p \rangle e^{iqx} \quad (\text{A.5})$$

Observe that the mean stress and mean momentum density satisfy exactly the overall equation of motion:

$$\nabla \langle \hat{\sigma} \rangle + i\omega \langle \hat{p} \rangle = 0 \quad (\text{A.6})$$

Define the effective mean displacement as

$$\langle \hat{u} \rangle(x) = \langle u \rangle e^{iqx}, \quad \langle u \rangle = \frac{1}{a} \int_{-a/2}^{+a/2} u(x) dx \quad (\text{A.7})$$

Then the effective mean strain and velocity are given by

$$\langle \hat{\epsilon} \rangle(x) = iq \langle \hat{u} \rangle(x), \quad \langle \hat{u} \rangle(x) = -i\omega \langle \hat{u} \rangle(x) \quad (\text{A.8})$$

Finally, the averaged stress–strain relation becomes

$$D^{\text{eff}} \langle \hat{\sigma} \rangle(x) = \langle \hat{\epsilon} \rangle(x) \quad (\text{A.9})$$

Based on Eqs. (A.5) and (A.8),  $D^{\text{eff}}$  is given by

$$D^{\text{eff}} = \frac{iq \langle u \rangle}{\langle \sigma \rangle} \quad (\text{A.10})$$

Similarly, the effective density ( $\rho^{\text{eff}}$ ) is given by

$$\rho^{\text{eff}} = \frac{\langle p \rangle}{-i\omega \langle u \rangle} \quad (\text{A.11})$$

Since  $\langle \hat{\sigma} \rangle$  and  $\langle \hat{p} \rangle$  satisfy the equation of motion (Eq. (A.6)), it is seen that the effective material parameters defined above automatically satisfy the dispersion relation:

$$\frac{1}{D^{\text{eff}} \rho^{\text{eff}}} = \frac{\omega^2}{q^2} \quad (\text{A.12})$$

## Appendix B. Explicit relations for matrices and proofs of identities

Eq. (28) is

$$\{\Sigma\} = \{\Phi\} \langle \sigma \rangle + \frac{1}{D_0} \{\Psi\} \langle \dot{u} \rangle$$

$$\{\dot{U}\} = \frac{1}{\rho_0} \{\Theta\} \langle \sigma \rangle + \{\Gamma\} \langle \dot{u} \rangle \quad (\text{B.1})$$

where

$$\{\Phi\} = \left[ -[\mathbf{A}_D] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_\rho]^{-1}[\mathbf{B}] \right]^{-1} \{f\}$$

$$\{\Psi\} = \frac{1}{\omega} \left[ -[\mathbf{A}_D] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_\rho]^{-1}[\mathbf{B}] \right]^{-1} [\mathbf{B}][\mathbf{A}_\rho]^{-1} \{f\}$$

$$\{\Theta\} = \frac{1}{\omega} \left[ -[\mathbf{A}_\rho] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_D]^{-1}[\mathbf{B}] \right]^{-1} [\mathbf{B}][\mathbf{A}_D]^{-1} \{f\}$$

$$\{\Gamma\} = \left[ -[\mathbf{A}_\rho] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_D]^{-1}[\mathbf{B}] \right]^{-1} \{f\}$$

$$[\mathbf{A}_D]_{\alpha\beta} = \bar{A}_{\alpha\beta} + \frac{f_{\alpha} D_0}{D_{\alpha} - D_0} \delta_{\alpha\beta}$$

$$[\mathbf{A}_\rho]_{\alpha\beta} = \bar{A}_{\alpha\beta} + \frac{f_{\alpha} \rho_0}{\rho_{\alpha} - \rho_0} \delta_{\alpha\beta}$$

$$\{f\}^T = \{f_1, f_2, \dots, f_{\bar{x}}\} \quad (\text{B.2})$$

The averaged consistency conditions (29) and (30), which are the final overall constitutive relations, the parameters  $\bar{D}$ ,  $\bar{\rho}$ ,  $\bar{S}_1$ , and  $\bar{S}_2$  can now be expressed as

$$\begin{aligned}\bar{D} &= D_0[1 - \{f\}^T \{\Phi\}] \\ \bar{S}_1 &= -D_0 \{f\}^T \frac{1}{D_0} \{\Psi\} = -\{f\}^T \{\Psi\} \\ \bar{S}_2 &= -\rho_0 \{f\}^T \frac{1}{\rho_0} \{\Theta\} = -\{f\}^T \{\Theta\} \\ \bar{\rho} &= \rho_0[1 - \{f\}^T \{\Gamma\}]\end{aligned}\tag{B.3}$$

### B.1. Mathematical structure of constitutive relations

We note that the matrices  $[\mathbf{A}_D]$ ,  $[\mathbf{A}_\rho]$ , and  $[\mathbf{B}]$ , given by

$$\begin{aligned}[\mathbf{A}_D]_{\alpha\beta} &= \sum_{\xi > 0} [f_\alpha g_\alpha(\xi) f_\beta g_\beta(-\xi) A(\xi) + f_\alpha g_\alpha(-\xi) f_\beta g_\beta(\xi) A(-\xi)] + \frac{f_\alpha D_0}{D_\alpha - D_0} \delta_{\alpha\beta} \\ [\mathbf{A}_\rho]_{\alpha\beta} &= \sum_{\xi > 0} [f_\alpha g_\alpha(\xi) f_\beta g_\beta(-\xi) A(\xi) + f_\alpha g_\alpha(-\xi) f_\beta g_\beta(\xi) A(-\xi)] + \frac{f_\alpha \rho_0}{\rho_\alpha - \rho_0} \delta_{\alpha\beta} \\ [\mathbf{B}]_{\alpha\beta} &= \sum_{\xi > 0} [f_\alpha g_\alpha(\xi) f_\beta g_\beta(-\xi) B(\xi) + f_\alpha g_\alpha(-\xi) f_\beta g_\beta(\xi) B(-\xi)]\end{aligned}\tag{B.4}$$

are Hermitian for real values of the wavenumber. For any Hermitian matrix  $[\mathbf{M}]$ , one has

$$[[\mathbf{M}]^{-1}]^* = [[\mathbf{M}]^*]^{-1} = [\mathbf{M}]^{-1}$$

where asterisk denotes a Hermitian transpose. Using this and the basic properties of transpose of products of matrices, it follows that both  $-[\mathbf{A}_D] + (1/v^2)[\mathbf{B}][\mathbf{A}_\rho]^{-1}[\mathbf{B}]^{-1}$  and  $-[\mathbf{A}_\rho] + (1/v^2)[\mathbf{B}][\mathbf{A}_D]^{-1}[\mathbf{B}]^{-1}$  are Hermitian matrices. Moreover, for a Hermitian matrix  $[\mathbf{M}]$ , it can be shown that the scalar  $\{f\}^T [\mathbf{M}] \{f\}$  is always real-valued. Therefore, from Eq. (B.2)

$$\begin{aligned}\bar{D} &= D_0 \left[ 1 - \{f\}^T \left[ -[\mathbf{A}_D] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_\rho]^{-1} [\mathbf{B}]^{-1} \right] \{f\} \right] \\ \bar{\rho} &= \rho_0 \left[ 1 - \{f\}^T \left[ -[\mathbf{A}_\rho] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_D]^{-1} [\mathbf{B}]^{-1} \right] \{f\} \right]\end{aligned}\tag{B.5}$$

are always real-valued. We also have the following identities:

$$\begin{aligned}\bar{S}_1^* &= -\{f\}^T \{\Psi\}^* \\ &= -\frac{1}{\omega} \left[ \{f\}^T \left[ -[\mathbf{A}_D] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_\rho]^{-1} [\mathbf{B}]^{-1} \right]^{-1} [\mathbf{B}][\mathbf{A}_\rho]^{-1} \{f\} \right]^* \\ &= -\frac{1}{\omega} \left[ [[\mathbf{B}][\mathbf{A}_\rho]^{-1} \{f\}]^* \left[ \{f\}^T \left[ -[\mathbf{A}_D] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_\rho]^{-1} [\mathbf{B}]^{-1} \right]^{-1} \right]^* \right] \\ &= -\frac{1}{\omega} \left[ \{f\}^T [\mathbf{A}_\rho]^{-1} [\mathbf{B}] \left[ -[\mathbf{A}_D] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_\rho]^{-1} [\mathbf{B}]^{-1} \right]^{-1} \{f\} \right] \\ &= -\frac{1}{\omega} \left[ \{f\}^T \left[ \left[ -[\mathbf{A}_D] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_\rho]^{-1} [\mathbf{B}]^{-1} \right] [\mathbf{B}]^{-1} [\mathbf{A}_\rho]^{-1} \right] \{f\} \right] \\ &= -\frac{1}{\omega} \left[ \{f\}^T \left[ -[\mathbf{A}_D][\mathbf{B}]^{-1} [\mathbf{A}_\rho] + \frac{1}{v^2} [\mathbf{B}]^{-1} \right] \{f\} \right]\end{aligned}\tag{B.6}$$

and

$$\begin{aligned}\bar{S}_2 &= -\{f\}^T \{\Theta\} \\ &= -\frac{1}{\omega} \left[ \{f\}^T \left[ -[\mathbf{A}_\rho] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_D]^{-1} [\mathbf{B}]^{-1} \right]^{-1} [\mathbf{B}][\mathbf{A}_D]^{-1} \{f\} \right]\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\omega} \left[ \{f\}^T \left[ [\mathbf{A}_D][\mathbf{B}]^{-1} \left[ -[\mathbf{A}_\rho] + \frac{1}{v^2} [\mathbf{B}][\mathbf{A}_D]^{-1}[\mathbf{B}] \right] \right]^{-1} \{f\} \right] \\
&= -\frac{1}{\omega} \left[ \{f\}^T \left[ -[\mathbf{A}_D][\mathbf{B}]^{-1}[\mathbf{A}_\rho] + \frac{1}{v^2} [\mathbf{B}] \right]^{-1} \{f\} \right]
\end{aligned} \tag{B.7}$$

which show that  $\overline{S}_1^* = \overline{S}_2$ . It can be seen that the matrices  $[\mathbf{A}_D]$ ,  $[\mathbf{A}_\rho]$ , and  $[\mathbf{B}]$  are Hermitian regardless of the asymmetric nature of the unit cell. While asymmetries related to dimensions affect the off diagonal terms in a way which preserves the Hermitian nature of the matrices, asymmetries related to material properties only affect the diagonal terms, thereby, preserving the Hermitian characteristics. This in effect means that, for any asymmetrical unit cell,  $\overline{D}$  and  $\overline{\rho}$  are always real-valued and that the coupling terms in (29) and (30) are always complex conjugates of each other.

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