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Harmonic Waves in Layered Composites: Bounds on Frequencies[1]

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Introduction

WAVES in elastic composites have been extensively studied in recent years [1–8].[4] In [7, 8], Nemat-Nasser proposed a new quotient which proved very effective for obtaining the eigenfrequencies of harmonic waves in composites with periodic structures. In contrast to the Rayleigh quotient which gives upper bounds [5], the new quotient provides neither lower nor upper bounds for the corresponding frequencies. In this Note, this difficulty is overcome by developing upper and lower bounds for the frequencies, using the results obtained from the new quotient.

New Quotient

Consider a layered elastic composite, and let the unit cell have the length 1. The eigenfrequencies are defined by the following eigenvalue problem:

$$D\sigma - u' = 0, \quad \sigma' + \rho\lambda u = 0, \quad \text{for } -\tfrac{1}{2} < x < \tfrac{1}{2}, \tag{1}$$

$$u(\tfrac{1}{2}) = u(-\tfrac{1}{2})e^{iQ}, \quad \sigma(\tfrac{1}{2}) = \sigma(-\tfrac{1}{2})e^{iQ}, \quad i = \sqrt{-1}, \tag{2}$$

where $D(x)$ and $\rho(x)$ are positive, piecewise continuous and continuously differentiable functions in $[-\frac{1}{2}, \frac{1}{2}]$ with the left and right-hand limits and derivatives; $\lambda = \omega^2$ is the squared frequency; $\sigma(x)e^{\pm i\omega t}$ and $u(x)e^{\pm i\omega t}$ are the stress and displacement, respectively; and Q is the wave number. The functions $D(x)$ and $\rho(x)$ admit a finite number of finite discontinuities.

The variational method developed in [7, 8] is equivalent to rendering stationary a new quotient

$$\lambda_N = \frac{\langle u', \sigma\rangle + \langle \sigma, u'\rangle - \langle D\sigma, \sigma\rangle}{\langle \rho u, u\rangle}, \quad \langle \rho u, v\rangle \equiv \int_{-1/2}^{1/2} \rho u v^* dx, \tag{3}$$

for *independent* fields σ and u which satisfy (2); in $(3)_2$ a superposed star denotes the complex conjugate. The corresponding Euler-Lagrange equations then are given by (1).

For approximate solution, one sets

$$u_M = \sum_{n=0}^{\pm M} U_n e^{i(Q+2\pi n)x},$$

$$\sigma_M = \sum_{n=0}^{\pm M} S_n e^{i(Q+2\pi n)x},$$

substitutes into (3), and rendering λ_N stationary obtains

$$\langle \sigma_M' + \rho\lambda_N u_M, e^{i(Q+2\pi n)x}\rangle = 0,$$

$$\langle D\sigma_M - u_M', e^{i(Q+2\pi n)x}\rangle = 0, \; n = 0, \pm 1, \ldots, \pm M, \tag{4}$$

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[4] Numbers in brackets designate References at end of Note.

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which give the first (approximate) $2M + 1$ eigenvalues and eigenfunctions.

Bounds

Assume that $D(x)$ and $\rho(x)$ are such positive functions that (1) and (2) admit a set of positive, distinct eigenvalues, $0 < \lambda_1 < \lambda_2 \ldots < \lambda_n < \ldots$, corresponding to the complete orthonormal set of eigenfunctions $\{\phi_n\}$, $\langle \rho\phi_n, \phi_m\rangle = \delta_{nm}$, δ_{nm} being the Kronecker delta. Set $\phi_n' = D\psi_n$, and note that $\psi_n' + \rho\lambda_n\phi_n = 0$ which shows that $\{\psi_n'/\lambda_n\}$ is also a complete orthonormal set of functions with the weighting function $R(x) = 1/\rho(x)$, i.e., $\langle R\psi_n', \psi_m'\rangle = \lambda_n\lambda_m\delta_{nm}$. It also follows that $(R\psi_n')' + D\lambda_n\psi_n = 0$ which shows that $\{\psi_n/\sqrt{\lambda_n}\}$ is a complete orthonormal set of functions with the weighting functions $D(x)$, i.e.,

$$\langle D\psi_n, \psi_m\rangle = \sqrt{\lambda_n\lambda_m}\,\delta_{nm}.$$

Set

$$u_M = \sum_{n=1}^{\infty} C_n\phi_n, \quad C_n = \langle \rho u_M, \phi_n\rangle,$$

$$\langle \rho u_M, u_M\rangle = \sum_{n=1}^{\infty} |C_n|^2 = 1, \quad (5)$$

$$\sigma_M = \sum_{n=1}^{\infty} A_n\psi_n, \quad A_n = \langle D\sigma_M, \psi_n\rangle/\lambda_n,$$

$$\langle D\sigma_M, \sigma_M\rangle = \sum_{n=1}^{\infty} \lambda_n|A_n|^2, \quad (6)$$

and observe that

$$\lambda_N = \langle D\sigma_M, \sigma_M\rangle = \langle u_M', \sigma_M\rangle = \sum_{n=1}^{\infty} \lambda_n|A_n|^2. \quad (7)$$

Simple calculation also shows that

$$G = \langle R\sigma_M', \sigma_M'\rangle = \sum_{n=1}^{\infty} \lambda_n^2|A_n|^2. \quad (8)$$

To obtain the desired bounds, observe that $(\lambda_n - \lambda_p) \times (\lambda_n - \lambda_{p+1}) \geq 0$ for every n and p. Hence

$$\lambda_n^2 - (\lambda_p + \lambda_{p+1})\lambda_n + \lambda_p\lambda_{p+1} \geq 0. \quad (9)$$

Multiply by $|A_n|^2$, and sum over n to obtain

$$G - (\lambda_p + \lambda_{p+1})\lambda_N + \lambda_p\lambda_{p+1}K \geq 0, \quad (10)$$

where

$$K = \sum_{n=1}^{\infty} |A_n|^2. \quad (11)$$

From (10) it immediately follows that, if $\bar{\lambda}_N = \lambda_N/K$ is such that[5] $\bar{\lambda}_N < \lambda_{p+1}$, then

$$\bar{\lambda}_N - \frac{\bar{G} - \bar{\lambda}_N^2}{\lambda_{p+1} - \bar{\lambda}_N} \leq \lambda_p, \quad (12)$$

where

$$\bar{G} = G/K, \ \bar{\lambda}_N = \lambda_N/K,$$

which defines a lower bound for the pth eigenvalue; note that the pth approximate λ_N, u_M, and σ_M must be used.

The quantity K can be obtained from

$$v_M = \Lambda + \int_{-1/2}^{x} D(\xi)\sigma_M(\xi)d\xi,$$

$$\Lambda = \int_{-1/2}^{1/2} D(\xi)\sigma_M(\xi)d\xi/(e^{iQ} - 1), \quad (13)$$

[5] The bound given by (12) is useless if this condition is not satisfied. This is a major weakness of the theory.

Table 1 Eigenfrequency ν and its lower and upper bounds for first two modes: $M = 1$, $\rho_2/\rho_1 = 3$, and $\eta_2/\eta_1 = 100$

Q	Lower bound (12) ν_l	New quotient ν_N	Upper bound (16) $\bar{\nu}_N$	Exact ν	Upper bound (15) ν_u	Rayleigh quotient $\bar{\nu}_R$
1.0	0.1932	0.1934	0.1933	0.1933	0.1999	0.4604
	1.3509	1.3548	1.3629	1.3616	1.3951	5.7947
2.0	0.3534	0.3541	0.3546	0.3544	0.3688	0.8541
	1.2796	1.2776	1.2886	1.2873	1.3204	5.1731
3.0	0.4317	0.4297	0.4341	0.4336	0.4510	1.0831
	1.2251	1.2462	1.2403	1.2386	1.3066	4.7414

which gives

$$K = \sum_{n=1}^{\infty} |A_n|^2 = \langle \rho v_M, v_M\rangle. \quad (14)$$

For upper bounds, one may start with $(\lambda_n - \lambda_{p-1})(\lambda_n - \lambda_p) \geq 0$ and obtain

$$\lambda_p \leq \bar{\lambda}_N + \frac{\bar{G} - \bar{\lambda}_N^2}{\bar{\lambda}_N - \lambda_{p-1}} \quad (15)$$

For λ_1, however, the Rayleigh quotient $\bar{\lambda}_R = \langle \eta u', u'\rangle/\langle \rho u, u\rangle$, $\eta = 1/D$, gives

$$\lambda_1 \leq \bar{\lambda}_R = \frac{\langle \eta v_M', v_M'\rangle}{\langle \rho v_M, v_M\rangle} = \frac{\langle D\sigma_M, \sigma_M\rangle}{K} = \frac{\lambda_N}{K} = \bar{\lambda}_N, \quad (16)$$

where $u = v_M$ is used.

Example

For illustration, consider a layered composite of two homogeneous constituents of equal length, and let ρ_1 and η_1 be the mass density and elasticity constant of material 1 occupying $-\frac{1}{2} \leq x \leq -\frac{1}{4}$ and $\frac{1}{4} \leq x \leq \frac{1}{2}$, and ρ_2 and η_2 those of material 2 occupying $|x| \leq \frac{1}{4}$. Set $\nu = \omega(\bar{\rho}/\bar{\eta})^{1/2}$, $\bar{\eta} = \frac{1}{2}(\eta_1 + \eta_2)$, and $\bar{\rho} = \frac{1}{2}(\rho_1 + \rho_2)$. The bounds obtained from (12), (15), and (16) are[6] shown in Table 1, together with the exact solutions and the results obtained from the Rayleigh quotient using the same approximate functions.

It is clear that the new quotient is very effective for the considered class of problems. Note that, although there are two sets of constants (namely, U_n's and S_n's) in the new quotient, only one set of equations needs to be considered, as is discussed in [7, 8]. This is because one set can be eliminated at the outset. Hence the work involved does *not* exceed that corresponding to the Rayleigh quotient. Note also that the bounds are sometimes even better than the new quotient from which they are obtained; i.e., the method gives bounds for the *exact* eigenvalues not for the new quotient.

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