Harmonic Waves in Layered Composites[1]

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Introduction

In a recent paper Kohn, Krumhansl, and Lee [1][3] have discussed the application of a variational method to harmonic waves in composites. This method leads to the usual Rayleigh quotient in which only the displacement field is given independent variation; these authors also discuss possible jump conditions in a special context, but do not use the corresponding expressions in their calculations (see [2] for further discussion). The same results, but with further application to harmonic waves in fiber-reinforced composites, can be found in a thesis by Wheeler [3]. An examination of the method presented in [1, 3] reveals that, unless a large number of terms are included in the approximate solutions, the numerical results are extremely poor.

The aim of this article is to show that, if the well-known Hellinger-Reissner variational method is modified for application to harmonic waves moving normal to layers in a layered composite, then the corresponding numerical results are astonishingly accurate, Fig. 1 and Tables 1 and 2.

Formulation

Consider a layered elastic composite whose properties vary normal to the layers (the x-axis) with the periodicity length a, so that $\rho(x + a) = \rho(x)$ and $D(x + a) = D(x)$, where ρ is the mass density, and $D(x) = 1/(\lambda + 2\mu)$, λ and μ being the Lamé constants. For harmonic waves moving normal to the layers, the stress and the displacement are in the form $\bar{\sigma} = \sigma(x)e^{i\omega t}$ and $\bar{u} = u(x)e^{i\omega t}$ and, according to the Floquet theory, satisfy

$$u\left(\frac{a}{2}\right) = u\left(-\frac{a}{2}\right)e^{iqa}, \quad \sigma\left(\frac{a}{2}\right) = \sigma\left(-\frac{a}{2}\right)e^{iqa}, \quad (1)$$

which are the quasi-periodicity conditions. In the variational method, these conditions can be accounted for by either the use of a Lagrangian multiplier or by simply calculating the work of the tractions at the two boundary points of a *representative cell*, as is shown later.

The equations of motion are

$$\frac{d\sigma}{dx} + \rho\omega^2 u = 0, \qquad D\sigma = \frac{du}{dx}, \quad (2)$$

Now observe that the variational functional must be real-valued, although the field quantities are complex-valued. Identifying the corresponding body forces by $\frac{1}{2}\omega^2 u$, hence write

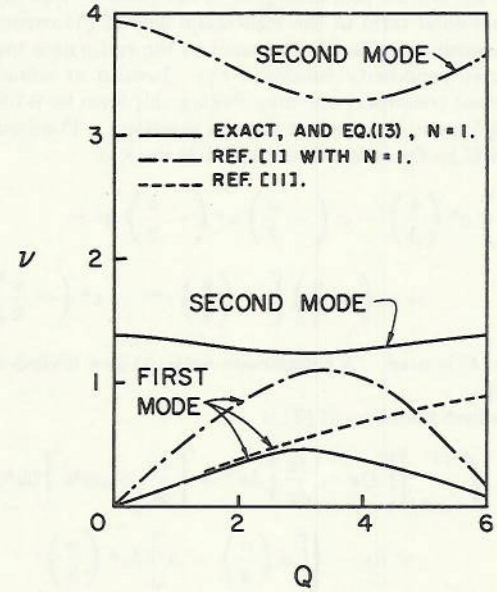

Fig. 1 Frequency parameter $\nu = a\omega(\bar{\rho}\bar{D})^{1/2}$ for $n_1 = n_2 = 1/2$, $\theta = 3$, $\gamma = 100$

Table 1 Eigenfrequency ν for first three modes: $N = 1$, $n_1 = n_2 = 1/2$, $\theta = 3$, $\gamma = 100$

Q	Exact ν	Present Method Eq. (9) ν	Present Method Eq. (9) % Error	Kohn, Krumhansl and Lee, Ref. [1] ν	Kohn, Krumhansl and Lee, Ref. [1] % Error
1.0	0.193	0.193	0.0	0.460	138
	1.362	1.355	-0.5	5.795	325
	2.497	2.715	8.7	7.597	204
2.0	0.354	0.354	0.0	0.854	141
	1.287	1.278	-0.7	5.173	302
	2.545	2.900	13.9	8.458	232

Table 2 Eigenfrequency ν for modes 2–5: $n_1 = n_2 = 1/2$, $\theta = 3$, $\gamma = 100$

Q	SECOND MODE, N = 2			THIRD MODE, N = 3		
	EXACT	Present Method Eq. (13)	Kohn, Krumhansl, and Lee, Ref. [1]	EXACT	Present Method Eq. (13)	Kohn, Krumhansl, and Lee, Ref. [1]
1.0	1.36	1.36	3.79	2.50	2.50	5.47
3.0	1.24	1.24	3.35	2.57	2.56	6.08
5.0	1.34	1.35	3.41	2.51	2.49	6.22

Q	FOURTH MODE, N = 4			FIFTH MODE, N = 5		
1.0	3.77	3.76	6.76	4.94	4.94	6.69
3.0	3.70	3.69	6.37	5.02	5.00	7.02
5.0	3.76	3.75	6.20	4.96	4.91	7.13

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[3] Numbers in brackets designate References at end of Note.

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$$I_1 = \int_{-a/2}^{a/2} \left\{ D\sigma\sigma^* + \rho\omega^2 uu^* - \sigma \frac{du^*}{dx} - \sigma^* \frac{du}{dx} \right\} dx + \left\{ \Lambda \left[u^*\left(\frac{a}{2}\right) - u^*\left(-\frac{a}{2}\right) e^{-iqa} \right] + \text{cc} \right\}, \quad (3)$$

where a superposed star denotes the complex conjugate, and cc stands for the complex conjugate of the quantity that precedes it.

The second term in the right-hand side of (3) corresponds to the constraint on the displacement at the end points imposed by the quasi-periodicity condition (1). Instead of using the displacement constraint one may replace this term by a term which corresponds to the work of the end tractions. The results, however, will be the same, since twice this work is

$$\sigma\left(\frac{a}{2}\right) u^*\left(\frac{a}{2}\right) - \sigma\left(-\frac{a}{2}\right) u^*\left(-\frac{a}{2}\right) + \text{cc} = \sigma\left(-\frac{a}{2}\right)\left[u^*\left(\frac{a}{2}\right) e^{iqa} - u^*\left(-\frac{a}{2}\right)\right] + \text{cc},$$

where (1) is used. A comparison with (3) now reveals that $\Lambda = \sigma(a/2)$.

The first variation of (3) is

$$\delta I_1 = \int_{-a/2}^{a/2} \left\{ \left[D\sigma - \frac{du}{dx} \right] \delta\sigma^* + \left[\frac{d\sigma}{dx} + \rho\omega^2 u \right] \delta u^* + \text{cc} \right\} \times dx - \left\{ \left[\sigma\left(\frac{a}{2}\right) - \Lambda \right] \delta u^*\left(\frac{a}{2}\right) - \left[\sigma\left(-\frac{a}{2}\right) - \Lambda e^{-iqa} \right] \delta u^*\left(-\frac{a}{2}\right) - \left[u\left(\frac{a}{2}\right) - u\left(-\frac{a}{2}\right) e^{iqa} \right] \delta\Lambda^* + \text{cc.} \right\} \quad (4)$$

It is seen that the vanishing of δI_1 for arbitrary variation of the indicated quantities guarantees the satisfaction of all the field equations and the quasi-periodicity conditions.

Let us now use (3) to calculate the dispersion curves for various harmonic modes of a composite consisting of layered elastic media bounded together. For simplicity, assume that a cell in this composite consists of two materials, M^β, $\beta = 1, 2$, where M^1 occupies the region $-a/2 \le x \le -b/2$ and $b/2 \le x \le a/2$, and M^2 occupies the region $-b/2 \le x \le b/2$. The mass density and the elastic constants of these materials will be denoted by ρ_β and D_β, $\beta = 1, 2$, respectively.

To apply (3), consider the test functions

$$u = \sum_{n=0}^{\pm N} \bar{U}_n e^{i(Q+2\pi n)\xi}, \qquad \sigma = \sum_{n=0}^{\pm N} \bar{S}_n e^{i(Q+2\pi)\xi}, \quad (5)$$

where $Q = qa$, $\xi = x/a$, and $\bar{U}_n$ and $\bar{S}_n$ are the Fourier coefficients which are to be calculated. Note that these test functions are *continuous* and hence no jump condition is necessary. Moreover, since the quasi-periodicity conditions are also satisfied, the last term in (3) drops out. Simple calculations then show that

$$I_1(N) = \sum_{\substack{n,m=0 \\ n\neq m}}^{\pm N} \left[\nu^2 \frac{(\theta-1)(n_1+\gamma n_2)}{n_1+\theta n_2} U_n U_m^* + \left(\frac{1-\gamma}{\gamma}\right) S_n S_m^* \right] \frac{\sin \pi(n-m)n_2}{\pi(n-m)} + \sum_{n=0}^{\pm N} \left[\nu^2(n_1+\gamma n_2) U_n U_n^* + i\left(\frac{n_1\gamma + n_2}{\gamma}\right)(Q+2\pi n)(S_n U_n^* - S_n^* U_n) \right], \quad (6)$$

where the following notation is used:

$$U_n = \bar{U}_n/\sqrt{D_1}a, \qquad S_n = \sqrt{D_1}\bar{S}_n, \qquad \nu = a\omega(\bar{\rho}/\bar{\eta})^{1/2},$$
$$\bar{\eta} = \frac{n_1}{D_1} + \frac{n_2}{D_2}, \qquad \bar{\rho} = n_1\rho_1 + n_2\rho_2, \qquad n_1 = \frac{a-b}{a}, \quad (7)$$
$$n_2 = \frac{b}{a}, \qquad \gamma = \frac{D_1}{D_2}, \qquad \theta = \frac{\rho_1}{\rho_2}.$$

Observe that for $N = 0$, equation (6) yields $\omega_0 = Q(\bar{\rho}\bar{D})^{1/2}$, $\bar{D} = n_1D_1 + n_2D_2$, which corresponds to the nondispersive result that can be obtained by the usual method of calculating the effective mass density and elastic modulus.

For $N \ge 1$, set

$$\mathbf{U} = \{U_{-N}, U_{-N+1}, \ldots, U_0, \ldots, U_N\}^T, \qquad \mathbf{S} = \{S_{-N}, S_{-N+1}, \ldots, S_0, \ldots, S_N\}^T,$$

where superposed T denotes the transpose, and express (6) as

$$I_1(N) = \begin{Bmatrix} \mathbf{U}^* \\ \mathbf{S}^* \end{Bmatrix} \begin{bmatrix} \mathbf{\Omega} & \mathbf{H} \\ \mathbf{H}^* & \mathbf{\Phi} \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \mathbf{S} \end{Bmatrix}, \quad (8)$$

where

$$\mathbf{\Omega} = \nu^2(\theta-1)\left(\frac{n_1+\gamma n_2}{n_1+\theta n_2}\right) \times \begin{bmatrix} \frac{n_1+\theta n_2}{\theta-1} & \frac{\sin \pi n_2}{\pi} & \frac{\sin 2\pi n_2}{2\pi} & \cdots \\ \frac{\sin \pi n_2}{\pi} & \frac{n_1+\theta n_2}{\theta-1} & \frac{\sin \pi n_2}{\pi} & \cdots \\ \frac{\sin 2\pi n_2}{2\pi} & \frac{\sin \pi n_2}{\pi} & \frac{n_1+\theta n_2}{\theta-1} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix},$$

$\mathbf{H} = i \operatorname{diag} \{Q - 2\pi N, Q + 2\pi(-N+1), \ldots, Q, \ldots, Q + 2\pi N\}$, and where the matrix $\mathbf{\Phi}$ is obtained by replacing in matrix $\mathbf{\Omega}$, $\nu^2(n_1 + \gamma n_2)$ by $n_1 + n_2/\gamma$, and $(\theta - 1)(n_1 + \gamma n_2)/(n_1 + \theta n_2)$ by $(1 - \gamma)/\gamma$, respectively. From the stationary condition imposed on (12), one arrives at $\mathbf{\Omega U} + \mathbf{HS} = \mathbf{0}$, $\mathbf{H^*U} + \mathbf{\Phi S} = \mathbf{0}$, which yield $\mathbf{S} = -\mathbf{H}^{-1}\mathbf{\Omega U}$ and $[\mathbf{H}^* - \mathbf{\Phi H}^{-1}\mathbf{\Omega}]\mathbf{U} = \mathbf{0}$. The characteristic equation then is

$$\det |\mathbf{H}^* - \mathbf{\Phi H}^{-1}\mathbf{\Omega}| = 0, \quad (9)$$

where $\mathbf{\Omega}$ is proportional to the eigenfrequency ν^2.

Numerical Results and Discussion

A significant feature for numerical calculation of equation (9) is that matrix $\mathbf{H}$ is diagonal, whereas, if the usual variational method in which only the displacement is varied (for example, the one discussed in [1]), is used, the characteristic equation takes on the form $\det |\mathbf{A} - \nu^2\mathbf{B}| = 0$ in which neither $\mathbf{A}$ nor $\mathbf{B}$ is diagonal.

For the first mode, our crudest approximation for which $N = 1$ (i.e., only three plane waves), yields extremely accurate results. Fig. 1 compares these results with those obtained from equations (50) and (51) of [1], and with those obtained by means of the "effective stiffness" method [4].

A remarkable conclusion obtained from (9) is that even the second and the third modes given here for $N = 1$, in which case (9) is only a 3×3 determinant, are quite accurate. To manifest this vividly, we have given in Table 1 the dimensionless eigenfrequency for the first three modes, together with the corresponding percentage errors, for the indicated values of the parameters.

As a comparison, this table also gives results obtained using the method proposed by Kohn, et al., [1].

For higher modes, N must be taken greater than 1. Table 2 gives results for modes 2 to 5 inclusive. As is seen, again these are very accurate.

In this Note, we have not discussed other general variational methods which may include discontinuities, in order to represent clearly the major features of the variational statement (3). The extension to the two and three-dimensional cases is also omitted, for the same reason. These and other new results are discussed in detail in [2].

References

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