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A fiber-bridged crack with rate-dependent bridging forces

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Abstract

A rate-dependent model is used to represent the bridging forces acting across a fiber-bridged slit crack in a homogeneous anisotropic elastic material, with the potential to characterize the inelastic behavior of bridged cracks in composites. The basic equilibrium equations are presented and solved, using a series of Chebychev polynomials and a suitable approximation numerical scheme. An example of a bridged crack in an isotropic homogeneous material is examined in detail. Results show that, under a constant external load, the stress intensity factors and the energy release rate increase with time until a critical time is reached, after which all these quantities approach the corresponding values for an unbridged crack.

Keywords: A. Fracture toughness; B. Anisotropic material; B. Viscoelastic material

1. Introduction

In fiber-reinforced composites (viewed as homogeneous anisotropic elastic solids), the fibers may bridge the existing or newly occurring cracks and therefore greatly increase the toughness of the material. Due to environmental effects, the bridging forces generated by the reinforcing fibers are often time-dependent. For example, for cracks in the fiber-reinforced polymer layers, the crack-bridging forces may vary with the time and may depend on the rate of change of the crack-opening-displacement, particularly under the action of moisture and temperature. In this paper, a model is proposed to represent the rate-dependent bridging forces in fiber-bridged slit cracks in anisotropic homogeneous elastic materials, where the bridging forces are proportional to the velocity of the crack-opening-displacement. Following Nemat-Nasser and Hori (1987),

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equilibrium equations are established. Then a solution strategy is proposed, using the Chebychev polynomials and a simple numerical integration scheme. The parameters involved in the equations are given explicitly for some of the often considered elastic materials. An example of a bridged crack in an isotropic elastic solid is worked out in detail. Results show that the stress intensity factors and the energy release rate increase in time, until a critical time, when all these quantities approach the corresponding values for an unbridged crack.

As a historical account, bridged cracks have been investigated by many researchers, e.g., Marshall et al. (1985), Marshall and Evans (1985), Budiansky (1986), Rose (1987), Swanson et al. (1987), Horii et al. (1987), Nemat-Nasser and Hori (1987), Hori and Nemat-Nasser (1990), Willis and Nemat-Nasser (1990), Willis (1993), and Movchan and Willis (1993, 1996, 1998). Using a different model, cracks in fiber-reinforced composites with rate-dependent bridging forces are studied by Gwo and Nair (1996), where the rate-dependent bridging forces are the result of the presence of a viscous fiber–matrix interfacial layer.

2. Equilibrium equations

Let $\hat{x}_k,\ k=1,2,3$, be a fixed Cartesian coordinate system. Consider a fiber-reinforced composite that consists of a homogeneous elastic matrix with embedded reinforcing fibers. The overall effects of the reinforcing fibers are represented through the overall anisotropy of the composite, as discussed, e.g., by Nemat-Nasser and Hori (1987). Hence, the elastic matrix is viewed as a homogeneous material. Consider a slit crack in such an elastic material, located on the $\hat{x}_1$-axis and extending from $\hat{x}_1=-c$ to $\hat{x}_1=c$ for $c>0$. The crack is bridged by the reinforcing fibers. Consider cases in which all variables are independent of $\hat{x}_3$. Assume that the crack is subjected to (possibly time-dependent) tractions, $\hat{\tau}_{2k}$, and rate-dependent bridging forces, $\hat{p}_k$, induced by the bridging fibers. The deformation is quasi-static, while all variables are parametrically time-dependent.

Using the superposition principle as in Nemat-Nasser and Hori (1987), equilibrium gives

$$\hat{\sigma}_{2k}+\hat{\tau}_{2k}+\hat{p}_k=0 \quad \text{for } k=1,2, \tag{1}$$

where $\hat{\sigma}_{2k}$ is the matrix-induced crack resistance. The expression for the crack resistance for a crack in a general anisotropic material is given in Nemat-Nasser and Hori (1987), and Ting (1996). In order to explicitly evaluate the involved elastic parameters, using the results given by Ni and Nemat-Nasser (1991, 1992, 2000), we consider the following expression for the crack resistance:

$$[\hat{\sigma}_{2k}]=\frac{\boldsymbol{\Lambda}_1}{\pi}\int_{-c}^{c}\frac{\hat{\boldsymbol{b}}(\xi,\hat{t})}{\hat{x}_1-\xi}\,\mathrm{d}\xi=\frac{\boldsymbol{\Lambda}_1}{\pi}\int_{-c}^{c}\frac{\hat{\boldsymbol{u}}(\xi,\hat{t})}{(\hat{x}_1-\xi)^2}\,\mathrm{d}\xi, \tag{2}$$

where $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{u}}$ are dislocation density and crack-opening-displacement vectors, respectively, and $\boldsymbol{\Lambda}_1$ is a constant matrix defined by

$$\boldsymbol{\Lambda}_1=-\tfrac{1}{2}[\mathrm{Im}(\boldsymbol{A}\boldsymbol{L}^{-1})]^{-1} \tag{3}$$

with matrices $\boldsymbol{A}$ and $\boldsymbol{L}$ given by

$$\boldsymbol{A} \equiv [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3], \quad \boldsymbol{L} \equiv [\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3]. \tag{4}$$

$[\boldsymbol{a}_j,\ \boldsymbol{l}_j]^{\mathrm{T}},\ j=1,2,3$, are the generalized eigenvectors, corresponding to three (possibly multiple) eigenvalues with positive imaginary parts, of the fundamental elasticity matrix $\boldsymbol{N}$ (Ingebrigtsen and Tonning, 1969; Ting, 1996),

$$\boldsymbol{N} \equiv \begin{bmatrix} \boldsymbol{n}_{11} & \boldsymbol{n}_{12} \\ \boldsymbol{n}_{21} & \boldsymbol{n}_{11}^{\mathrm{T}} \end{bmatrix}, \tag{5}$$

where

$$\boldsymbol{n}_{11} \equiv -\boldsymbol{T}^{-1}\boldsymbol{R}^{\mathrm{T}}, \quad \boldsymbol{n}_{21} \equiv \boldsymbol{Q} - \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{\mathrm{T}}, \quad \boldsymbol{n}_{12} \equiv -\boldsymbol{T}^{-1} \tag{6}$$

and where $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{T}$ are defined in terms of the elasticity tensor, C_{ijkl}, of the matrix material by

$$\boldsymbol{Q} \equiv [C_{j1k1}], \quad \boldsymbol{R} \equiv [C_{j1k2}], \quad \boldsymbol{T} \equiv [C_{j2k2}]. \tag{7}$$

Note that the matrix $\boldsymbol{A}\boldsymbol{L}^{-1}$ is unique, although the normalization and order of numbering in (4) for $[\boldsymbol{a}_j, \boldsymbol{l}_j]^{\mathrm{T}},\ j=1,2,3$, are not specified.

When only the in-plane components of the field variables are at focus, then matrix $\boldsymbol{\Lambda}_1$ reduces to

$$\boldsymbol{\Lambda}_1 = \begin{bmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{bmatrix}. \tag{8}$$

The components of this matrix are functions of the elastic constants of the solid. As examples, in what follows explicit expressions for $\alpha_j,\ j=1,3$, are summarized for isotropic, orthotropic, and monoclinic elastic solids (Suo, 1990; Ni and Nemat-Nasser, 1991, 1992, 2000; Ting, 1992, 1996).

2.1. Isotropic solids

$$\alpha_1 = \alpha_2 = \frac{2\mu}{1+\kappa}, \quad \alpha_3 = 0, \tag{9}$$

where μ is the shear modulus; and $\kappa = 3-4v$ for plane strain, and $\kappa = (3-v)/(1+v)$ for plane stress, with v being the Poisson ratio.

2.2. Orthotropic solids

Let the x_1, x_2, and x_3 axes be coincident with the material symmetry directions. Then,

$$\alpha_1 = \frac{(c_{12}+c_0)}{2}\left[\frac{c_{66}(c_0-c_{12})}{c_{22}(2c_0-c)}\right]^{1/2}, \quad \alpha_2 = \alpha_1\left[\frac{c_{22}}{c_{11}}\right]^{1/2}, \tag{10}$$

where c_0 and c are defined by

$$c_0 \equiv (c_{11}c_{22})^{1/2}, \quad c \equiv c_0 - c_{12} - 2c_{66}. \tag{11}$$

Here, c_{ij} are the components of the elasticity matrix $[c_{ij}]$, $i,j=1,2,4,5,6$; see, e.g., Nemat-Nasser and Hori (1993).

2.3. Monoclinic solids

Assume that the x_1, x_2-plane is the only plane of symmetry of the material. Then, it follows that

$$\alpha_1 = \mathrm{Im}[p_1 p_2(\bar{p}_1 + \bar{p}_2)]/e, \quad \alpha_2 = \mathrm{Im}[p_1 + p_2]/e, \quad \alpha_3 = -\mathrm{Im}[p_1 p_2]/e \tag{12}$$

with

$$e = 2s_{11}\{\mathrm{Im}[p_1 + p_2]\,\mathrm{Im}[p_1 p_2(\bar{p}_1 + \bar{p}_2)] - (\mathrm{Im}[p_1 p_2])^2\}, \tag{13}$$

where the compliance matrix $[s_{ij}]$, $i,j = 1,2,4,5,6$, is the inverse of the elasticity matrix $[c_{ij}]$, and p_1 and p_2 are the eigenvalues (with positive imaginary parts) of the fundamental elasticity matrix $\boldsymbol{N}$; these are found to be the roots of the equation

$$s_{11}p^4 - 2s_{16}p^3 + (2s_{12} + s_{66})p^2 - 2s_{26}p + s_{22} = 0. \tag{14}$$

Assume now that the bridging forces, $\hat{p}_k$, are proportional to the time-rate of change of the crack-opening-displacement, $\hat{u}_k$, as follows:

$$\hat{p}_k = -K_0\hat{g}(\hat{x}_1,\hat{t})\frac{\partial \hat{u}_k}{\partial t}(\hat{x}_1,\hat{t}), \tag{15}$$

where K_0 describes the reference strength of the bridging force, having the physical dimension of force multiplied by time divided by volume, and the nondimensional function $\hat{g}$ represents the distribution of the bridging forces.

Introduce the following dimensionless variables:

$$x = \hat{x}_1/c, \quad t = \hat{t}/T_0, \quad b_k(x,t) = \hat{b}_k(\hat{x}_1,\hat{t}), \tag{16}$$

$$v_k(x,t) = \hat{u}_k(\hat{x}_1,\hat{t})/c, \quad g(x) = \hat{g}(\hat{x}_1), \tag{17}$$

$$\boldsymbol{\Lambda}_{01} = \boldsymbol{\Lambda}_1/\alpha_0, \quad \tau_{2k}(x,t) = \hat{\tau}_{2k}(\hat{x}_1,\hat{t})/\alpha_0, \tag{18}$$

where c is half the crack length, T_0 is the material time scale, and $\alpha_0 = \sqrt{\alpha_1\alpha_2 - \alpha_3^2}$. The equilibrium equations become

$$\frac{(\boldsymbol{\Lambda}_{01})_{kr}}{\pi}\int_{-1}^{1}\frac{v_r(\xi,t)}{(x-\xi)^2}\,\mathrm{d}\xi - lg(x)\frac{\partial}{\partial t}v_k(x,t) = -\tau_{2k}(x,t) \tag{19}$$

with boundary and initial conditions

$$v_k(\pm 1,t) = 0, \quad v_k(x,0) = 0, \tag{20}$$

respectively, where $k,r = 1,2$, repeated indices are summed, and the nondimensional parameter $l = c/c_0$ measures the crack length in terms of a length scale, c_0, defined by

$$c_0 = \left[\frac{\alpha_0 T_0}{K_0}\right] \tag{21}$$

which is a material parameter. In (19), $l \gg 1$ corresponds to a 'long crack', and $l \ll 1$, to a 'short crack'.

3. Solution method

Solve Eqs. (19) incrementally in time. Using the Euler method, replace the time derivative by the difference

$$\frac{\partial}{\partial t}v_k(x,t) \simeq \frac{1}{\Delta t}[v_k(x,t) - v_k(x,t-\Delta t)] \tag{22}$$

for a suitable time increment Δt that ensures a desired accuracy, as discussed later on. Then Eqs. (19) become

$$\frac{(\boldsymbol{\Lambda}_{01})_{kr}}{\pi}\int_{-1}^{1}\frac{v_r(\xi,t)}{(x-\xi)^2}\,\mathrm{d}\xi - l\frac{g(x)}{\Delta t}v_k(x,t) = -\tau_{2k}(x,t) - l\,\frac{g(x)}{\Delta t}v_k(x,t-\Delta t). \tag{23}$$

These are the same type of equations as those considered in Nemat-Nasser and Hori (1987). Hence, the Chebychev polynomial-based Galerkin method can be applied. Consider cases in which $g(x)$, $\tau_k(x,t)$, and the unknown functions $v_k(x,t)$ are all sufficiently smooth, so that they can be expanded in terms of the Chebychev polynomials, as follows:

$$g(x) = \sum_{n=0}^{\infty} g_n U_n(x), \quad \tau_{2k}(x,t) = \sum_{n=0}^{\infty}\chi_n^{(k)}(t)U_n(x), \tag{24}$$

$$v_k(x,t) = W(x)\sum_{n=0}^{\infty}\phi_n^{(k)}(t)U_n(x), \tag{25}$$

where $W(x) = \sqrt{1-x^2}$ is the weight function, and the Chebychev polynomials of the second kind are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \theta = \cos^{-1}x. \tag{26}$$

Expression (23) can now be reduced to a system of linear equations, using the above expansions and the following identities for the Chebychev polynomials (Luke, 1969):

$$\int_{-1}^{1}\frac{\mathrm{d}\xi}{(x-\xi)^2}W(\xi)U_n(\xi) = -\pi(n+1)U_n(x),$$

$$\int_{-1}^{1}W(\xi)U_n(\xi)U_m(\xi)\,\mathrm{d}\xi = \frac{\pi}{2}\delta_{nm}. \tag{27}$$

This results in,

$$(\boldsymbol{\Lambda}_{01})_{kr}\phi_m^{(r)}(t) + \frac{l}{\Delta t}\sum_{n=0}^{\infty}\hat{B}_{mn}\phi_n^{(k)}(t) = \psi_m^{(k)}(t), \quad \phi_m^{(k)}(0) = 0 \tag{28}$$

for $m,n = 0,1,\ldots,$ where

$$\psi_m^{(k)}(t) = \chi_m^{(k)}/(m+1) + \frac{l}{\Delta t}\,\hat{B}_{mn}\phi_n^{(k)}(t-\Delta t), \tag{29}$$

$$\hat{B}_{mn} = \frac{2}{\pi(m+1)}\sum_{s=0}^{\infty} A_{mns} g_s \tag{30}$$

and A_{mnr} are defined by

$$A_{nmr} = \begin{cases} 0, & m+n+r \text{ odd}, \\ \frac{4(m+1)(n+1)(r+1)}{(m+n-r+1)(m-n+r+1)(-m+n+r+1)(m+n+r+3)}, & m+n+r \text{ even}. \end{cases} \tag{31}$$

As considered in Nemat-Nasser and Hori (1987), system (28) may be approximated by the finite system

$$(\boldsymbol{\Lambda}_{01})_{kr}\phi_m^{(r)}(t) + \frac{l}{\Delta t}\sum_{n=0}^{N} B_{mn}\phi_n^{(k)} = \psi_m^{(k)}(t), \tag{32}$$

$$B_{mn} = \frac{2}{\pi(m+1)}\sum_{s=0}^{N} A_{mns} g_s. \tag{33}$$

Note that

$$\boldsymbol{\Lambda}_{01} = \boldsymbol{E}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\boldsymbol{E}^{-1} \tag{34}$$

with $\boldsymbol{E} = [\boldsymbol{w}_1, \boldsymbol{w}_2]$, and $(\boldsymbol{\Lambda}_{01})\boldsymbol{w}_r = \lambda_r \boldsymbol{w}_r$ for $r = 1, 2$.

From (32), $\phi_m^{(k)}(t)$, $k = 1, 2$, $m = 0, 1, \ldots, N$, is obtained,

$$\phi_m^{(k)}(t) = E_{ks}\left[\lambda_k \boldsymbol{I} + \frac{l}{\Delta t}\boldsymbol{B}\right]_{mn}^{-1} E_{sr}^{-1}\psi_n^{(r)}(t), \tag{35}$$

where $\boldsymbol{B}$ is the matrix corresponding to (33), repeated indices are summed for $s = 1, 2$, and $n = 0, 1, \ldots, N$.

Combining (35) with (29), it follows that

$$\phi_m^{(k)}(t) = E_{ks}[\lambda_k \boldsymbol{I}\Delta t + l\boldsymbol{B}]_{mn}^{-1} E_{sr}^{-1}\left[\frac{\chi_n^{(r)}\Delta t}{n+1} + lB_{np}\phi_p^{(r)}(t-\Delta t)\right], \quad \phi_n^{(k)}(0) = 0. \tag{36}$$

The eigenvalues λ_k, $k = 1, 2$, and the matrix $\boldsymbol{E}$ can be evaluated explicitly as follows.

Case (I): $\alpha_3 \neq 0$.

This case includes the general anisotropic materials, e.g., the monoclinic materials with the $x_3 = 0$-plane as its only plane of symmetry. Then, the eigenvalues λ_k, $k = 1, 2$, of $\boldsymbol{\Lambda}_{01}$ are

$$\lambda_1 = \frac{1}{2\alpha_0}[\alpha_1 - \alpha_2 - \Delta], \quad \lambda_2 = \frac{1}{2\alpha_0}[\alpha_1 - \alpha_2 + \Delta], \tag{37}$$

where $\alpha_0 = \sqrt{\alpha_1\alpha_2 - \alpha_3^2}$ and $\Delta = \sqrt{(\alpha_1 - \alpha_2)^2 + 4\alpha_3^2}$. The eigenvector matrix $\boldsymbol{E}$ is given by

$$\boldsymbol{E} = \begin{bmatrix} \alpha_1 - \alpha_2 - \Delta & \alpha_1 - \alpha_2 + \Delta \\ 2\alpha_3 & 2\alpha_3 \end{bmatrix}. \tag{38}$$

Case (II): $\alpha_3 = 0$.

This case corresponds to the isotropic material, or the orthotropic material with the x_1, x_2, x_3 axes as the material symmetry directions. Then, $\boldsymbol{\Lambda}_{01}$ is a diagonal matrix, and the eigenvalues are

$$\lambda_1 = \sqrt{\alpha_1/\alpha_2}, \quad \lambda_2 = \sqrt{\alpha_2/\alpha_1}. \tag{39}$$

The eigenvector matrix $\boldsymbol{E}$ is an identity matrix.

Comments on solution method: The proposed solution algorithm involves truncating infinite sums in (28). The convergence of the solution obtained by this approximation can be examined by considering an equivalent system of Cauchy singular integral equations of the second kind with the dislocation density as the unknown function. For the latter, the convergence of the polynomial-based Galerkin algorithm is well known. For simplicity, consider a single equation corresponding to system (23),

$$\frac{1}{\pi}\int_{-1}^{1}\frac{v(\xi)}{(x-\xi)^2}\,\mathrm{d}\xi - lg(x)v(x) = \psi(x), \quad v(\pm 1) = 0. \tag{40}$$

Since $b(x) = -(\partial/\partial x)v(x)$, (40) becomes

$$\frac{1}{\pi}\int_{-1}^{1}\frac{b(\xi)}{x-\xi}\,\mathrm{d}\xi + lg(x)\int_{-1}^{x} b(\xi)\,\mathrm{d}\xi = -\psi(x). \tag{41}$$

From

$$v(x) = \sqrt{1-x^2}\sum_{n=0}^{\infty}\phi_n U_n(x), \tag{42}$$

it follows that

$$b(x) = \frac{1}{\sqrt{1-x^2}}\sum_{n=0}^{\infty}(n+1)\phi_n T_n(x) \equiv \frac{1}{\sqrt{1-x^2}}\,p(x), \tag{43}$$

where $T_n(x)$ are the Chebychev polynomials of the first kind, and, in the derivation, the following relation is used (Luke, 1969):

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sqrt{1-x^2}U_n(x)] = -\frac{(n+1)T_{n+1}(x)}{\sqrt{1-x^2}}. \tag{44}$$

Now, (41) becomes

$$\frac{1}{\pi}\int_{-1}^{1}\frac{p(\xi)}{\sqrt{1-\xi^2}(x-\xi)}\,\mathrm{d}\xi + lg(x)\int_{-1}^{x}\frac{p(\xi)}{\sqrt{1-\xi^2}}\,\mathrm{d}\xi = -\psi(x), \tag{45}$$

which can be rewritten as

$$\boldsymbol{H}_{1/w}\,p + \boldsymbol{K}p = -\psi, \tag{46}$$

where $\boldsymbol{H}_{1/w}$ and $\boldsymbol{K}$ denote the corresponding integral operators.

Consider the Hilbert space L^2_w,

$$L^2_w = \left\{ f\colon [-1,1] \to R; \int_{-1}^{1} w(x) f^2(x)\,\mathrm{d}x < \infty \right\} \tag{47}$$

with the inner product

$$(f,g)_w = \int_{-1}^{1} w(x) f(x) g(x)\,\mathrm{d}x. \tag{48}$$

Then, the convergence in $L^2_{1/w}$ of the Galerkin approximate solution follows directly from the fact that $\boldsymbol{K}$ is compact from space $L^2_{1/w}$ to L^2_w; and, $\boldsymbol{H}_{1/w} + \boldsymbol{K}$ is injective (Mikhlin and Prödorf, 1986; Golberg, 1990).

Calculation of physical quantities: The physical quantities are now expressed in terms of the Chebychev polynomials as follows.

3.1. Crack-opening-displacement

From (25), the nondimensional crack-opening-displacement is expressed in terms of a series of Chebychev polynomials,

$$v_k(x,t) = \sum_{n=0}^{N} \phi_n^{(k)}(t) U_n(x) \sqrt{1-x^2} \tag{49}$$

for $k = 1, 2$, where again, $U_n(x)$ is the Chebychev polynomial of the second kind.

3.2. Dislocation density

The dislocation density over $|x| \leqslant 1$ is obtained from the crack-opening-displacement,

$$b_k(x,t) = -\frac{\partial}{\partial x} v_k(x,t) = \sum_{n=0}^{N} (n+1) \phi_n^{(k)}(t) \frac{T_n(x)}{\sqrt{1-x^2}} \tag{50}$$

for $k = 1, 2$.

3.3. Stresses for $|x| > 1$

The stresses on the plane $x_2 = 0$ for $|x| > 1$ are expressed as

$$\sigma_{2k} = \frac{(\boldsymbol{\Lambda}_{01})_{kr}}{\pi} \int_{-1}^{1} \frac{b_r(\xi,t)}{x-\xi}\,\mathrm{d}\xi + \tau_{2k}(x,t), \tag{51}$$

where $|x| = |x_1|/c > 1$. In view of the expansion (50), the last equation is written as

$$\sigma_{2k} = \frac{(\boldsymbol{\Lambda}_{01})_{kr}}{\pi} \sum_{n=0}^{N} \phi_n^{(r)}(t)(n+1) \int_{-1}^{1} \frac{T_{n+1}(\xi)}{(x-\xi)\sqrt{1-\xi^2}}\,\mathrm{d}\xi. \tag{52}$$

Making use of the relation (Luke, 1969)

$$\int_{-1}^{1} \frac{T_{n+1}(\xi)}{(x-\xi)\sqrt{1-\xi^2}}\,\mathrm{d}\xi = \pi\left[\frac{T_n(x)}{\sqrt{x^2-1}} - U_{n+1}(x)\right], \tag{53}$$

for $x > 1$, from (52), the stresses for $x > 1$ are given by

$$\sigma_{2k} = (\Lambda_{01})_{kr}\sum_{n=0}^{N}\phi_n^{(r)}(t)(n+1)\left[\frac{T_n(x)}{\sqrt{x^2-1}} - U_{n+1}(x)\right] + \tau_{2k}(x,t), \tag{54}$$

where the Chebychev polynomials of the first or second kind for $x > 1$, are understood as

$$T_n(x) = \frac{n!\sqrt{\pi}}{\Gamma(n+1/2)}P_n^{(-1/2,-1/2)}(x), \tag{55}$$

$$U_n(x) = \frac{(n+1)!\sqrt{\pi}}{2\Gamma(n+3/2)}P_n^{(1/2,1/2)}(x), \tag{56}$$

with the Jacobi polynomials for $x > 1$ defined by

$$P_n^{(a,b)}(x) = \frac{(x-1)^{-a}(x+1)^{-b}}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(x-1)^{n+a}(x+1)^{n+b}]. \tag{57}$$

3.4. Stress-intensity-factors

Define the stress-intensity-factors for Modes I and II as

$$K_{\mathrm{I}} = \lim_{x\to 1}\sigma_{22}(x)\sqrt{2\pi(x-1)} \tag{58}$$

and

$$K_{\mathrm{II}} = \lim_{x\to 1}\sigma_{12}(x)\sqrt{2\pi(x-1)}, \tag{59}$$

respectively. Then, from expression (54) for the stresses, there follow the stress-intensity-factors for Modes I and II,

$$K_{\mathrm{I}} = \sqrt{\pi}\sum_{n=0}^{N}(\Lambda_{01})_{2r}\phi_n^{(r)}(t)(n+1) \tag{60}$$

and

$$K_{\mathrm{II}} = \sqrt{\pi}\sum_{n=0}^{N}(\Lambda_{01})_{1r}\phi_n^{(r)}(t)(n+1), \tag{61}$$

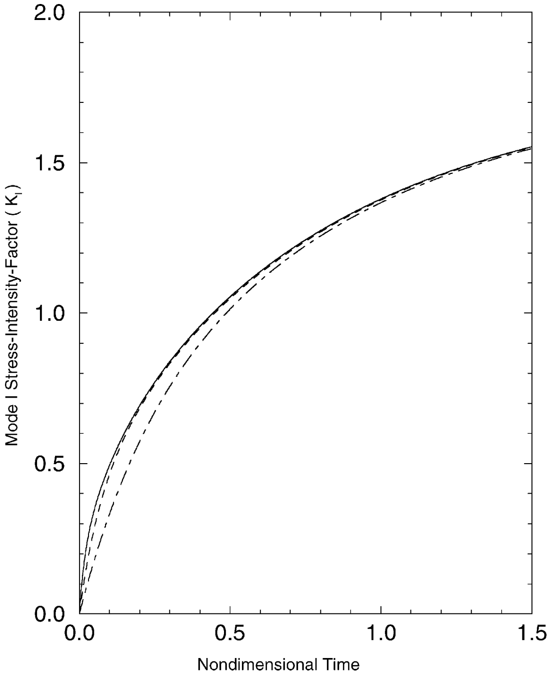

Fig. 1. Nondimensional mode I stress-intensity-factor $K_{\rm I}$ vs. the nondimensional time $\hat{t}/T_0$ for a bridged crack in an isotropic material, under uniform external load; $l = \pi(1+\kappa)K_0c/2\mu T_0 = 1$, $\Delta t = 0.001$, and $N+1=$ 4 (dot–dashed), 8 (dashed), and 16, 32, 48, and 64 (solid) curves, respectively.

respectively, which can be rewritten as

$$K_{\rm I} = \sqrt{\pi}/\alpha_0[\alpha_3\Phi_1 + \alpha_2\Phi_2] \tag{62}$$

and

$$K_{\rm II} = \sqrt{\pi}/\alpha_0[\alpha_1\Phi_1 + \alpha_3\Phi_2], \tag{63}$$

where Φ_k, $k = 1, 2$, are defined by

$$\Phi_k = \sum_{n=0}^{N} \phi_n^{(k)}(t)(n+1). \tag{64}$$

3.5. Energy-release-rate

The energy-release-rate near the right crack tip, $x = 1$, is defined by

$$\mathscr{E} = \lim_{\delta\to 0} \frac{1}{2\delta} \int_0^{\delta} \sigma_{2i}(1+\eta)v_i(1-(\delta-\eta))\,\mathrm{d}\eta. \tag{65}$$

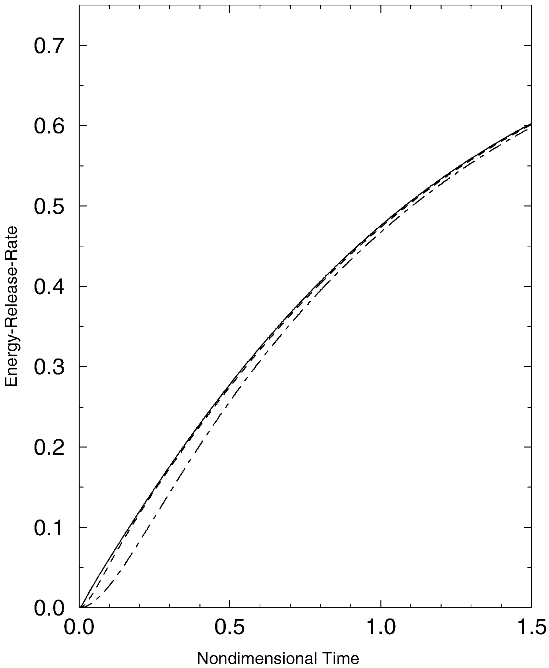

Fig. 2. Nondimensional energy-release-rate $\mathscr{E}$ vs. the nondimensional time $\hat{t}/T_0$ for a bridged crack in an isotropic material, under uniform external load; $l = \pi(1+\kappa)K_0c/2\mu T_0 = 1$, $\Delta t = 0.001$, and $N + 1 = 4$ (dot–dashed), 8 (dashed), and 16, 32, 48, and 64 (solid) curves, respectively.

In view of the above results, the energy-release-rate is calculated to be

$$\mathscr{E} = \frac{\pi}{4\alpha_0}[\alpha_1\Phi_1^2 + 2\alpha_3\Phi_1\Phi_2 + \alpha_2\Phi_2^2], \tag{66}$$

where Φ_k, $k = 1,2$, are defined in (64).

4. Example

As an example, consider an isotropic material. Then, $\alpha_3 = 0$, and matrix $\boldsymbol{\Lambda}_{01}$ and its eigenvector matrix $\boldsymbol{E}$ are both identity matrices. The equilibrium equations (19) now decouple,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{v_k(\xi,t)}{(x-\xi)^2}\,\mathrm{d}\xi - lg(x)\frac{\partial}{\partial t}v_k(x,t) = -\tau_{2k}(x,t) \tag{67}$$

for $k = 1,2$. Employing the Chebychev-polynomial-based Galerkin method, described in the last section, and expansions (24) and (25), Eq. (36) becomes

$$\phi_m^{(k)}(t) = [\lambda_k\Delta t\boldsymbol{I} + l\boldsymbol{B}]_{mn}^{-1}\left[\frac{\chi_n^{(k)}}{n+1}\Delta t + lB_{np}\phi_p^{(k)}(t-\Delta t)\right], \quad \phi_m^{(k)}(0) = 0 \tag{68}$$

with $k = 1,2$, $m,n = 0,1,\dots,N$, and repeated indices are summed for $n,p = 0,1,\dots,N$.

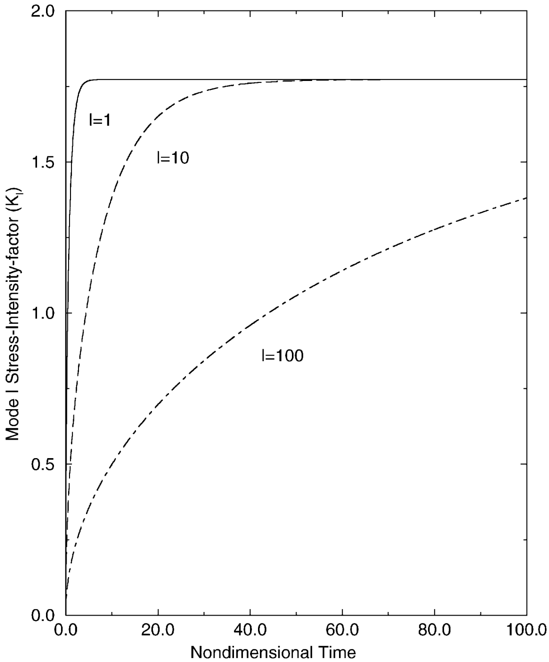

Fig. 3. Dependence of the nondimensional mode I stress-intensity-factor on the nondimensional time for a bridged crack in an isotropic material, under uniform external load and indicated values of $l = \pi(1+\kappa)K_0 c/2\mu T_0$.

Assume now that the applied tractions, $\tau_{2k}(x,t)$, and the bridging distribution function, $g(x)$, are uniform,

$$g(x) = g_0, \quad \tau_{2k}(x,t) = \chi_0^k \equiv \tau_{2k}, \quad \chi_n^k = 0, \quad \text{for } n \neq 0 \tag{69}$$

for $k = 1, 2$, and note that

$$(\boldsymbol{B})_{mn} = \frac{2g_0}{\pi(m+1)} A_{mn0}, \tag{70}$$

$$A_{mn0} = \begin{cases} 0, & m+n \text{ odd}, \\ \frac{4(m+1)(n+1)}{(m+n+1)(m-n+1)(-m+n+1)(m+n+3)}, & m+n \text{ even} \end{cases} \tag{71}$$

for $0 \leqslant m, n \leqslant N$.

The stress-intensity-factors and the energy-release-rate are now given by

$$K_{\mathrm{I}} = \sqrt{\pi}\tau_{22}\Phi, \tag{72}$$

$$K_{\mathrm{II}} = \sqrt{\pi}\tau_{12}\Phi \tag{73}$$

and

$$\mathscr{E} = \frac{\pi}{4}[\tau_{12}^2 + \tau_{22}^2]\Phi^2, \tag{74}$$

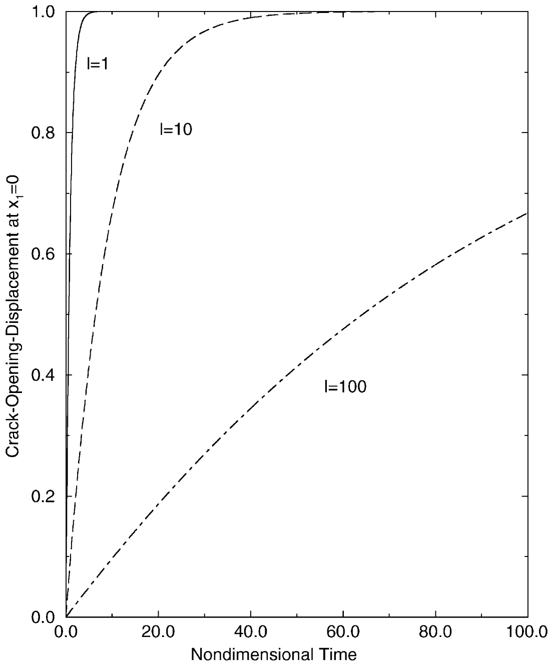

Fig. 4. Dependence of the nondimensional crack-opening-displacement at $x = 0$ on the nondimensional time for a bridged crack in an isotropic material, under uniform external load and indicated values of $l = \pi(1 + \kappa)K_0c/2\mu T_0$.

respectively, where Φ is defined by

$$\Phi = \sum_{n=0}^{N} \phi_n(t)(n + 1). \tag{75}$$

The crack-opening-displacement at $x = 0$ is given by

$$v(0, t) = \sum_{n=0}^{N} \phi_n(t)U_n(0), \quad U_n(0) = \sin\frac{n + 1}{2}\pi. \tag{76}$$

Numerical calculations are performed to estimate the stress-intensity-factor $K_{\rm I}(t)$, the crack-opening-displacement $v_2(0, t)$, and the energy-release-rate $\mathscr{E}(t)$, for an uniform external load defined by $[\tau_{21} = 0,\ \tau_{22} = 1]$, for $g(x) = 1$ and the nondimensional crack length $l = (1 + \kappa)K_0c/(2\mu T_0) = 1, 10, 100$.

The convergence of the resulting solution is checked numerically in two steps: (i) numerical evaluation of the effect of the time-step size on the solution; and (ii) numerical evaluation of the truncation errors. The three physical quantities, $K_{\rm I}(t)$, $v_2(0, t)$, and $\mathscr{E}(t)$ are evaluated for different time steps, $\Delta t = 0.005, 0.001, 0.0005$, and different number of terms in the finite sums, i.e., 4, 8, 16, 32, 48, and 64 terms, and the results are compared. All three time steps yield essentially the same results, and hence $\Delta t = 0.001$ is used in producing the final data.

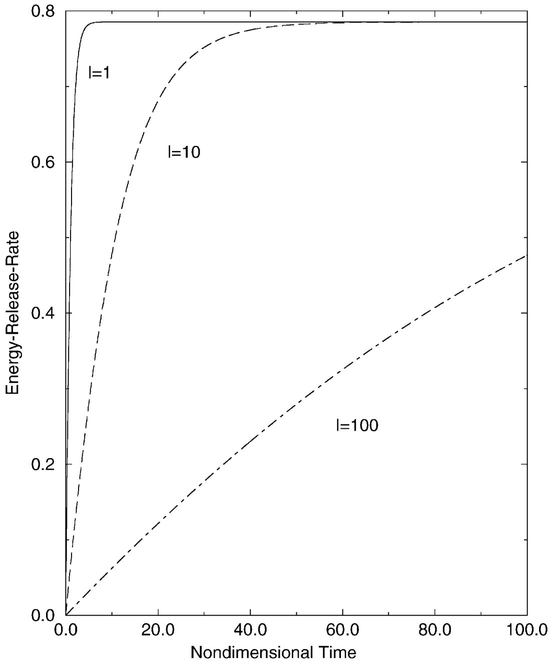

Fig. 5. Dependence of the nondimensional energy-release-rate $\mathscr{E}$ on the nondimensional time for a bridged crack in an isotropic material, under uniform external load and indicated values of $l = \pi(1+\kappa)K_0c/2\mu T_0$.

The calculated stress-intensity-factor and energy-release-rate for $l=1$ and $\Delta t=0.001$ are given in Figs. 1, and 2, respectively. The dot–dashed and dashed curves correspond to $N=3$ and 7, respectively, while the solid curves represent results for $N=15$, 31, 47, and 63, which yield basically the same results.

For the nondimensional crack length, $l=(1+\kappa)K_0c/(2\mu T_0) \equiv 1, 10, 100$, the results of the stress-intensity-factor, the crack-opening-displacement at $x=0$, and the energy-release-rate, are presented in Figs. 3, 4, and 5, respectively, where N is taken to be 31, and $\Delta t=0.001$.

These numerical results suggest that, when the rate-dependent bridging forces are defined by (15), as the time increases, the stress-intensity-factor, the crack-opening -displacement, and the energy-release-rate all increase accordingly until a critical time is reached, at which they approach the limiting values that are the corresponding quantities for the case of the crack without bridging. The value of such a critical time depends on the nondimensional crack length. Noting that the nondimensional crack length is defined by $l=K_0c/(\alpha_0 T_0)$, the value of the critical time is greater when the bridging strength is greater, or the elastic modulus, characterized by α_0, is smaller.

The numerical results presented for illustration are, however, typical. Solutions for other more general cases of applied stresses can be developed by a linear combination of these typical results. Note that the dimensional stress-intensity-factors are

$$\hat{K}_{\mathrm{I}} = \alpha_0\sqrt{c}K_{\mathrm{I}}, \quad \hat{K}_{\mathrm{II}} = \alpha_0\sqrt{c}K_{\mathrm{II}}, \tag{77}$$

and the energy-release-rate is

$$\hat{\mathscr{E}} = \alpha_0 c \mathscr{E}. \tag{78}$$

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