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Alternative solution methods for crack problems in plane anisotropic elasticity, with examples

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Abstract

Two-dimensional crack problems of homogeneous, anisotropic, linear elasticity are solved using the *Riemann–Hilbert* method. To this end, the *Riemann–Hilbert* problem of line-discontinuity is formulated for *anisotropic* plane problems and the necessary parameters and functions are identified. For illustration, the method is applied to obtain the complete stress field and the stress intensity factors for a crack in an infinite anisotropic plate which is loaded on a part of one of its faces. Then, the well-established method of *continuously distributed edge-dislocations* is considered and illustrated via some example problems; e.g., an infinite anisotropic plate under uniform farfield loads containing: 1. a closed frictional crack and a pair of arbitrarily-located single edge-dislocations, and 2. an infinite row of equally-spaced parallel open cracks.

The illustrative examples reveal that the first method offers an effective solution technique for problems where unbalanced tractions are applied on crack surfaces, whereas for problems with *self-equilibrating* loads applied on the crack faces, the second method is generally well suited. In addition, the method of *resultant forces along the crack* is discussed and its formulation in terms of the dislocation density functions and also the *crack-opening displacements* (which is new) is presented. The solutions to some of the example problems are provided in some detail, and for others, just the key formulae (e.g., stress functions and stress intensity factors) are calculated and analyzed. In brief, this paper presents the generalization of the Riemann–Hilbert method from isotropic to anisotropic in-plane elasticity problems, and also provides a collection of certain basic two-dimensional anisotropic crack problems; some of the results here are also new. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Anisotropic elasticity; Fracture; Hilbert method; Crack; Dislocations; Green function; Resultant force; Half-space; Integral equation; Stress intensity factor; Periodic cracks; Frictional crack; Dislocated crack; Continuous distribution of edge-dislocations

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1. Introduction

To simplify the analysis, isotropic elasticity is usually assumed in many fracture mechanics problems. Electronic packaging and devices usually include strongly anisotropic materials; this is especially the case of single crystals. While anisotropy introduces additional material parameters (see, e.g. Simmons, 1971 for elastic constants of different single crystals), it does render the basic field equations better structured and hence, simpler to solve. A large class of analytical solutions is based on the fundamental work of Muskhelishvili (1953), followed by Savin (1961), and Lekhnitskii (1963); see Sneddon (1961) for a survey of the earlier work. For isotropic materials, crack problems have been extensively studied theoretically, numerically, and experimentally; see, e.g., Erdogan and Sih (1963), Liebowitz (1968), Bilby and Eshelby (1968), Rice (1968, 1972), and Sih (1973). In contrast, less work has been done for cracks in anisotropic solids, some of which are: Eshelby et al. (1953), Sih et al. (1965), Willis (1966), Tsai and Wu (1971), Barnett and Asaro (1972), Wu (1974), Delale and Erdogan (1977), Hoenig (1982), Nemat-Nasser and Hori (1987), Sham and Zhou (1989), Miller and Stock (1989), Obata et al. (1989), Suo (1990), Ni and Nemat-Nasser (1991), Gao and Chiu (1992), Azhdari (1995), and Azhdari and Nemat-Nasser (1996a, 1996b, 1998). Section 21 of the book by Nemat-Nasser and Hori (1993), and a comprehensive book by Ting (1996) are among the more recent general accounts of anisotropic elasticity, emphasizing linear fracture mechanics. Finally, analytical/numerical and numerical solution methods such as the weight function, finite-element, and boundary-element methods are studied by many researchers; for a review and references, see, e.g., Aliabadi and Rooke (1991).

In this paper, we examine two solution methods for cracks in anisotropic planes. The first solution method is based on the Hilbert problem, as generalized by Obata et al. (1989), providing analytical solutions for cracks subjected to unbalanced prescribed tractions on their faces. The second solution method is the well-known continuously distributed dislocations (CDD) technique; see a comprehensive book by Weertman (1996). Moreover, in the context of the second method, the resultant-force (in contrast to the traction) method on the crack line is also discussed. In addition, a collection of solutions of cracks in two-dimensional anisotropic planes is provided (see the last paragraph of this section).

View a crack as a line in a plane, across which some physical quantities may admit jump discontinuities. This line has upper and lower faces, and the boundary conditions for the crack can be prescribed on these faces independently. Thus, a crack can be formulated as a Hilbert problem. This problem and its application are well established for isotropic plane problems; see Muskhelishvili (1953) and Savin (1961). However, its application to the anisotropic cases has received less attention. Sih and Liebowitz (1968) applied the Hilbert problem to anisotropic planes containing line discontinuities. A general formulation (for loading as well as material symmetry) is considered in the present work. In Section 3, the complete formulation and the corresponding parameters are presented for both traction- and displacement-boundary conditions. Note that this solution method (i.e., formulating a crack as a Hilbert problem) applies even when the crack surfaces are subjected to different boundary conditions. Problems of this nature have been solved using different methods such as conformal mapping; see, e.g., Savin (1961) and Bowie (1973).

Next, consider modeling a crack by a continuously distributed dislocations along its line (CDD method); see the original works by Stroh (1958, 1962), Willis (1970), and a recent book by Hills et al. (1996). Note that the CDD method is used for other applications, such as, modeling the crack-tip plasticity; see, e.g., Atkinson and Kanninen (1977) and Horii and Nemat-Nasser (1986). For modeling a crack, the CDD method involves the following steps: 1) modeling the discontinuity across the crack surface by using a continuous distribution of edge dislocations, $b(s) = b_x(s) + ib_y(s)$, where s measures length along the crack line; 2) formulating the corresponding stress field; 3) calculating the tractions leading to equilibrium integral equations; and 4) solving the resulting system of integral equations for the unknown $b(s)$, often, numerically by a collocation method. Note that the above steps are based on the superposition principle; see, e.g. Bueckner (1958).

The crack-surface boundary conditions, can alternatively be formulated by considering the *resultant forces on the crack surfaces* (in contrast to *crack-surface tractions*). The primary unknown in these methods can be the dislocation density or the crack opening displacement (COD); each may be formulated using either the crack-surface tractions or the crack-surface resultant-forces. Thus, the CDD method may fall into four categories; for references to these four methods, see, e.g. Lo (1978), Cheung and Chen (1987), Kaya and Erdogan (1987), and Azhdari (1995), respectively. Note that the CDD method is also referred to as integral transforms/continuous dislocations method.

This paper is organized as follows. First, in Section 2, a summary of the basic equations of anisotropic linear elasticity, required for the solution of fracture problems is provided. In Section 3, the Hilbert method is generalized for application to anisotropic elasticity problems. As an illustration, the problem of a crack partially loaded on its upper surface (unbalanced boundary conditions) is solved in Section 3.1. This solution is then extended to the case of a fully loaded upper face, and to the case of a pair of concentrated forces applied at an arbitrary point on the upper face (Sections 3.2 and 3.3). Then, the results of Sections 3.1, 3.2 and 3.3 are modified to obtain the corresponding solution for a half-plane. Section 4 deals with the CDD method which is illustrated by 5 crack problems with balanced loads on their faces; those are as follows. Section 4.1 considers the Green functions for an open crack with one (or a pair of centrally symmetric) arbitrarily-located edge-dislocation. The same problems, but for a closed crack with frictional and cohesive interface, are considered in Section 4.2. Section 4.3 addresses the problem of a crack dislocated at one end, with and without farfield loads, including a partially closed frictionless case. In Section 4.4, the problem of an open crack partially loaded by self-equilibrating tractions on its faces is solved, including the case of fully loaded faces, as well as when concentrated forces are applied to an arbitrary point on both faces. Section 4.5 focuses on an infinite row of periodic parallel open cracks, under uniform farfield loads (tension and shear). Then, the resulting coupled singular integral equations are numerically solved. An approximate formula is also given for estimating the stress intensity factors (SIFs). Section 5 considers the method of the resultant force, in terms of both the dislocation density function and the crack-opening displacement, as the primary unknowns. Finally, to render the paper self-contained, necessary topics are included in Appendices A–D. Most of the formulation and results of Sections 3, 4.2, 4.3, 4.5 and 5 are new.

2. Introductory formulation of anisotropic elasticity

Consider plane problems in anisotropic, homogeneous, linearly elastic solids. For the conditions where the in-plane and out-of-plane deformations decouple, the strain–stress relations in the x, y, z-coordinate system (Fig. 1) are

$$\varepsilon_{xx} = C_{11}\sigma_{xx} + C_{12}\sigma_{yy} + C_{13}\sigma_{zz} + C_{16}\sigma_{xy}, \tag{2.1a}$$

$$\varepsilon_{yy} = C_{12}\sigma_{xx} + C_{22}\sigma_{yy} + C_{23}\sigma_{zz} + C_{26}\sigma_{xy}, \tag{2.1b}$$

$$\varepsilon_{zz} = C_{13}\sigma_{xx} + C_{23}\sigma_{yy} + C_{33}\sigma_{zz} \tag{2.1c}$$

and

$$\gamma_{xy} = C_{16}\sigma_{xx} + C_{26}\sigma_{yy} + C_{66}\sigma_{xy}, \ (\gamma_{xy} = 2\varepsilon_{xy}), \tag{2.1d}$$

where $C_{ij} = C_{ji}$; i, j = 1, 2,..., 6, are the relevant elements of the compliance matrix of the material in the x, y-coordinate system; see Appendix A for the relationship between the C_{ij}'s and the engineering material constants. Note that for plane-stress conditions ($\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$), Eqs. (2.1a), (2.1b), (2.1c) and (2.1d) is applicable, whereas for plane-strain conditions ($\varepsilon_{zx} = \varepsilon_{zy} = \varepsilon_{zz} = 0$), the C_{ij}'s of Eqs. (2.1a), (2.1b), (2.1c) and (2.1d) should be replaced by $C_{ij} - C_{i3}C_{j3}/C_{33}$.

Savin (1961) and Lekhnitskii (1963) have shown that the problems of two-dimensional anisotropic elasticity can be conveniently formulated in terms of two independent analytic functions, $\phi(z_1)$ and $\psi(z_2)$. The complex variables, z_1 and z_2, are

$$z_j = x + s_j y, \tag{2.2a}$$

with

$$s_j = \alpha_j + i\beta_j, \quad \text{where } \beta_j > 0, \quad j = 1, 2. \tag{2.2b}$$

The parameters s_1 and s_2 are the roots of the characteristic equation (derived from the compatibility equation),

$$C_{11}s^4 - 2C_{16}s^3 + (2C_{12} + C_{66})s^2 - 2C_{26}s + C_{22} = 0. \tag{2.3}$$

Due to the positive-definiteness of the elastic energy, the characteristic equation has either complex or purely imaginary roots which are pairwise each other's complex conjugate, i.e.,

$$s_3 = \bar{s}_1, \quad s_4 = \bar{s}_2 \Longrightarrow z_3 = \bar{z}_1, \quad z_4 = \bar{z}_2; \tag{2.4}$$

without loss of generality, we choose s_1 and s_2 such that their imaginary parts are positive; see Eq. (2.2b). In Appendix A, the relations among C_{mn}, s_j, α_j and β_j are given. For the isotropic case, $s_1 = s_2 = i = \sqrt{-1}$, and the above formulation ceases to hold; the solution is valid only for $s_1 \neq s_2$. Nevertheless, by selecting $s_1 = (1 + \varepsilon)i$ and $s_2 = (1 - \varepsilon)i$ ($\varepsilon \ll 1$), the anisotropic formulation can be applied to the isotropic case as well; this is accomplished by setting, e.g. $E_{11}/E_{22} = 1.000001$ (see Appendix A).

The stress, displacement, and the resultant force fields then are

$$\sigma_{xx} = 2 \text{ Real} \left[s_1^2 \Phi(z_1) + s_2^2 \Psi(z_2)\right], \tag{2.5a}$$

$$\sigma_{yy} = 2 \text{ Real} \left[\Phi(z_1) + \Psi(z_2)\right], \tag{2.5b}$$

$$\sigma_{xy} = -2 \text{ Real} \left[s_1 \Phi(z_1) + s_2 \Psi(z_2)\right], \tag{2.5c}$$

$$u_x = 2 \text{ Real} \left[p_1 \phi(z_1) + p_2 \psi(z_2)\right], \quad u_y = 2 \text{ Real} \left[q_1 \phi(z_1) + q_2 \psi(z_2)\right], \tag{2.5d}$$

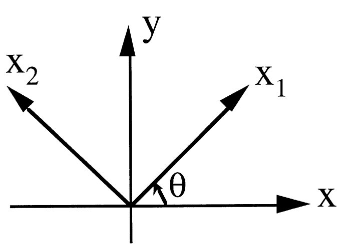

Fig. 1. Body and material coordinate systems.

$$f_x = 2\ \text{Real}\,[s_1\phi(z_1) + s_2\psi(z_2)] + c_x \quad \text{and} \quad f_y = -2\ \text{Real}\,[\phi(z_1) + \psi(z_2)] + c_y, \tag{2.5e}$$

where

$$p_i = C_{11}s_i^2 + C_{12} - C_{16}s_i, \quad q_i = C_{12}s_i + \frac{C_{22}}{s_i} - C_{26}, \tag{2.6}$$

$\Phi(z) = \phi'(z)$, $\Psi(z) = \phi'(z)$ and c_x and c_y are the resultant-force constants to be determined.

If the coordinate system x–y is rotated (counter-clockwise) by an angle ω to a new coordinate system ζ–η, then the transformed potential functions will be

$$\hat{\Phi}(z_1) = (\cos\omega + s_1\sin\omega)^2\Phi(z_1) \quad \text{and} \quad \hat{\Psi}(z_2) = (\cos\omega + s_2\sin\omega)^2\Psi(z_2). \tag{2.7a}$$

Moreover, the stresses in the new coordinate system are (see Azhdari, 1995)

$$\sigma_{\zeta\zeta} = 2\ \text{Real}\Big[\hat{s}_1^2\hat{\Phi}(z_1) + \hat{s}_2^2\hat{\Psi}(z_2)\Big] = 2\ \text{Real}\big[\Phi(z_1)L_1 + \Psi(z_2)L_2\big], \tag{2.8a}$$

$$\sigma_{\eta\eta} = 2\ \text{Real}\big[\hat{\Phi}(z_1) + \hat{\Psi}(z_2)\big] = 2\ \text{Real}\big[\Phi(z_1)M_1 + \Psi(z_2)M_2\big] \tag{2.8b}$$

and

$$\sigma_{\zeta\eta} = -2\ \text{Real}\big[\hat{s}_1\hat{\Phi}(z_1) + \hat{s}_2\hat{\Psi}(z_2)\big] = 2\ \text{Real}\big[\Phi(z_1)N_1 + \Psi(z_2)N_2\big], \tag{2.8c}$$

where

$$L_j = (s_j\cos\omega - \sin\omega)^2, \quad M_j = (\cos\omega + s_j\sin\omega)^2, \tag{2.9a}$$

$$N_j = (\cos\omega + s_j\sin\omega)(\sin\omega - s_j\cos\omega) \quad \text{and} \quad \hat{s}_j = \frac{s_j\cos\omega - \sin\omega}{\cos\omega + s_j\sin\omega}. \tag{2.9b}$$

Note that Eq. (2.9b) is the transformation formula for the roots s_j of Eq. (2.3).

Within most parts of this work, cracks are modeled as a continuous distribution of edge dislocations along their lines. For that, the solution of a single edge-dislocation $b^0 = (b_x^0, b_y^0)$ located at an arbitrary point $z^0 = (x^0, y^0)$ in an infinitely extended plate is given below. Such a dislocation generates a stress field at a general point (x, y) with the following potential functions; see Obata et al. (1989):

$$\Phi^{\mathrm{D}}(x, y; x_0, y_0) = \frac{1}{2\pi i C_{11}}\frac{s_1 b_x^0 - b_y^0}{\tilde{s}_1}\frac{1}{z_1 - z_1^0} \tag{2.10a}$$

and

$$\Psi^{\mathrm{D}}(x, y; x_0, y_0) = \frac{1}{2\pi i C_{11}}\frac{s_2 b_x^0 - b_y^0}{\tilde{s}_2}\frac{1}{z_2 - z_2^0}, \tag{2.10b}$$

where

$$\tilde{s}_1 = (s_1 - s_2)(s_1 - s_3)(s_1 - s_4) \quad \text{and} \quad \tilde{s}_2 = (s_2 - s_1)(s_2 - s_3)(s_2 - s_4). \tag{2.10c}$$

Note that $z_j^0 = x^0 + s_j y^0$ and $\tilde{s}_3$, $\tilde{s}_4$, z_3^0 and z_4^0 are complex conjugates of $\tilde{s}_1$, $\tilde{s}_2$, z_1^0 and z_2^0, respectively. See Appendix C for comments on single edge dislocation and dislocation density functions.

3. *Riemann–Hilbert* problems of line-discontinuity and cracks in anisotropic plates

In two dimensional elasticity, a crack is viewed as an arc across which the displacement field is suitably discontinuous. On each of the crack faces, concentrated or distributed forces may act. The aim then is to determine the corresponding elastic field and the crack tip SIFs. For this, various methods can be used, ranging from the classical mapping-function and Hilbert techniques (Muskhelishvili, 1953 and Savin, 1961) to the analytical/numerical weight-function method (Bueckner, 1970 and Rice, 1972). For a plane containing straight cuts, Muskhelishvili (1953) has formulated the corresponding boundary-value problem in terms of the Hilbert problem. For anisotropic elasticity problems, Sih and Liebowitz (1968) were the first to mention the Riemann–Hilbert method (denoted as '*Hilbert*', hereafter), though their formulation is applicable to only restricted cases. In what follows, we give a general formulation to solve a problem of line discontinuity by the Hilbert method, within the framework given by Obata et al. (1989). In order to clearly present the problem and the solution, first, the definition of a Hilbert problem is stated and its corresponding solution for a single discontinuity in a plane is given. Then, for a plane problem of an anisotropic medium, we perform the required generalization such that this solution is cast into a Hilbert problem framework.

To illustrate the Hilbert method, a crack in an infinitely extended anisotropic plate, loaded on a part of its upper face, is solved; for this simple example, the method of this section provides a complete stress field as well as the SIFs. This solution, along with the examples of the method of *continuously distributed dislocations* given in Section 4, shows that when the applied loads on the crack faces are not balanced, then formulation of the problem by the Hilbert method seems to be more effective among other possible analytical techniques. However, when traction-boundary conditions are symmetric or anti-symmetric (balanced tractions), the method of continuously distributed dislocations may be used to solve the corresponding boundary-value problem. At the end of this section, it is shown that the solution of the example problem can be reduced to the problem of an anisotropic *half plane*; by letting the crack length go to infinity, the upper half of the medium reduces to a half plane loaded on the plane $y = 0$.

Consider the following case of the Hilbert problem. For a given function, $g(s)$, defined on an arc L with end points a and b, find a complex-valued potential, $G(z)$, such that:

$$G^{+}(s) - \tau G^{-}(s) = g(s) \quad \text{for } s \in L, \tag{3.1}$$

where $G^{+}(s) = \lim_{z^{+}\to s} G(z)$, $G^{-}(s) = \lim_{z^{-}\to s} G(z)$ and τ is a given constant. The general solution of Eq. (3.1), which is holomorphic on the plane except for the line L, singular at the end points of L, and decaying to zero at infinity, is then given by Muskhelishvili (1953) as

$$G(z) = \frac{X(z)}{2\pi i}\left\{\int_L \frac{g(t)}{X^{+}(t)(t-z)}\,\mathrm{d}t + P_0\right\}, \tag{3.2a}$$

where

$$X(z) = (z-a)^{-\xi}(z-b)^{\xi-1}, \quad \xi = \frac{1}{2\pi i}\log(\tau), \tag{3.2b}$$

$$X^{+}(t) = \lim_{z^{+}\to t} X(z) \quad \text{and} \quad P_0 = \text{constant}. \tag{3.2c}$$

Next, considering Eqs. (3.2a–c), we formulate the Hilbert problem for line discontinuities in an anisotropic medium.

Referring to Eqs. (2.5a–e), the stresses and displacements are expressed in terms of two stress functions, ϕ and ψ, and their derivatives, Φ and Ψ, as

$$\sigma_{yy} = \Phi(z_1) + \Psi(z_2) + \overline{\Phi(z_1)} + \overline{\Psi(z_2)}, \tag{3.3a}$$

$$\sigma_{xy} = -s_1\Phi(z_1) - s_2\Psi(z_2) - s_3\overline{\Phi(z_1)} - s_4\overline{\Psi(z_2)}, \tag{3.3b}$$

$$u_x = p_1\phi(z_1) + p_2\psi(z_2) + p_3\overline{\phi(z_1)} + p_4\overline{\psi(z_2)} \tag{3.3c}$$

and

$$u_y = q_1\phi(z_1) + q_2\psi(z_2) + q_3\overline{\phi(z_1)} + q_4\overline{\psi(z_2)}. \tag{3.3d}$$

The stresses and displacements can be combined to form alternative expressions (these are more suited to the formulation of the Hilbert problem) as follows:

$$\sigma_{yy} - \alpha\sigma_{xy} = (1+\alpha s_1)\Theta(z_1) + (1+\alpha s_3)\Omega(z_3) + (1+\alpha s_2)\{\Psi(z_2) - \Psi(z_1)\} + (1+\alpha s_4)\left\{\overline{\Psi(z_2)} - \overline{\Psi(z_1)}\right\}, \tag{3.4a}$$

$$u_x + \alpha u_y = (p_1+\alpha q_1)\theta(z_1) + (p_3+\alpha q_3)\omega(z_3) + (p_2+\alpha q_2)\{\psi(z_2) - \psi(z_1)\} + (p_4+\alpha q_4)\left\{\overline{\psi(z_2)} - \overline{\psi(z_1)}\right\}, \tag{3.4b}$$

with

$$\Theta(z) = \theta'(z), \quad \Omega(z) = \omega'(z), \quad \Phi(z) = \phi'(z), \quad \Psi(z) = \psi'(z), \tag{3.5}$$

$$\Omega(z) = \bar{\Phi}(z) + \frac{1+\alpha s_4}{1+\alpha s_3}\bar{\Psi}(z), \quad \Theta(z) = \Phi(z) + \frac{1+\alpha s_2}{1+\alpha s_1}\Psi(z), \tag{3.6a}$$

$$\omega(z) = \bar{\phi}(z) + \frac{p_4+\alpha q_4}{p_3+\alpha q_3}\bar{\psi}(z), \quad \theta(z) = \phi(z) + \frac{p_2+\alpha q_2}{p_1+\alpha q_1}\psi(z) \tag{3.6b}$$

and the parameter α is the root of the quadratic function

$$\Delta_3\alpha^2 + 2\Delta_2\alpha - \Delta_1 = 0 \tag{3.7}$$

(the Δ_ks are defined in Appendix A).

Note that the newly defined auxiliary functions Θ and Ω naturally inherit the holomorphic properties of Φ and Ψ. From Appendix A and Eq. (3.7), a more explicit form for α is as follows:

$$\alpha = \frac{-\Delta_2 \pm i\sqrt{\Delta_0}}{\Delta_3} = \frac{-(s_1 s_2 - s_3 s_4) \pm \sqrt{(s_1 - s_3)(s_1 - s_4)(s_2 - s_3)(s_2 - s_4)}}{s_1 s_2 (s_3 + s_4) - s_3 s_4 (s_1 + s_2)}$$

$$= \frac{-(\alpha_1 \beta_2 + \alpha_2 \beta_1) \mp i\sqrt{\beta_1 \beta_2 \left((\alpha_1 - \alpha_2)^2 + (\beta_1 + \beta_2)^2\right)}}{\beta_1 \left(\alpha_2^2 + \beta_2^2\right) + \beta_2 \left(\alpha_1^2 + \beta_1^2\right)}; \tag{3.8}$$

note that $\alpha \to i$ for isotropic materials. Moreover, combination of Eqs. (3.8), (2.6) results in the following identities:

$$\frac{1 + \alpha s_2}{1 + \alpha s_1} = \frac{p_2 + \alpha q_2}{p_1 + \alpha q_1} = -\frac{1 + \bar{\alpha} s_2}{1 + \bar{\alpha} s_1} = -\frac{p_2 + \bar{\alpha} q_2}{p_1 + \bar{\alpha} q_1}. \tag{3.9}$$

Since s_1 and s_2 are distinctive characteristic roots with positive imaginary parts (see Section 2), one deduces that:

1. the term under the square-root bracket is always positive,
2. α *cannot* be a real number (for orthotropic materials for which $\Delta_2 = 0$, α is purely imaginary; see also Appendix A).

Thus, we can determine a unique set of Ω and Θ in terms of Φ and Ψ and vice versa; the relations among different stress functions can be obtained by consideration of Eqs. (3.4a), (3.4b), (3.5), along with identities of Eq. (3.9). Thus, Eqs. (3.6a,b), yield the following expressions:

$$\Phi(z) = \frac{1}{2}(\Theta(z) + \bar{\Omega}(z)), \quad \Psi(z) = \frac{1}{2}\frac{1 + \alpha s_1}{1 + \alpha s_2}(\Theta(z) - \bar{\Omega}(z)), \tag{3.10a}$$

$$\phi(z) = \frac{1}{2}(\theta(z) + \bar{\omega}(z)) \quad \text{and} \quad \psi(z) = \frac{1}{2}\frac{p_1 + \alpha q_1}{p_2 + \alpha q_2}(\theta(z) - \bar{\omega}(z)). \tag{3.10b}$$

Note that because of Eq. (3.9), the constant coefficients in Eqs. (3.6a,b) and (3.10a,b) can be written in alternative forms. For example, any of the four expressions in Eq. (3.9) could be used as the constant coefficient in Eq. (3.10a) or Eq. (3.10b).

Now, consider the complex variable $z_k = x + (\alpha_k + i\beta_k)y$ and its conjugate $\bar{z}_k = x + (\alpha_k - i\beta_k)y$. On the x-axis, where y may approach zero from the upper side (+) or the lower side (–), Eq. (3.4a) takes on the form

$$\sigma_{yy}^{+} - \alpha\sigma_{xy}^{+} = (1 + \alpha s_1)\Theta^{+}(x) + (1 + \alpha s_3)\Omega^{-}(x), \tag{3.11a}$$

and

$$\sigma_{yy}^{-} - \alpha\sigma_{xy}^{-} = (1 + \alpha s_1)\Theta^{-}(x) + (1 + \alpha s_3)\Omega^{+}(x). \tag{3.11b}$$

Note that the rest of the terms in Eqs. (3.4a) and (3.4b) vanish as y goes to zero from either side of the x-axis. Adding and subtracting both sides of Eqs. (3.11a) and (3.11b), we obtain

$$[(1 + \alpha s_1)\Theta(x) + (1 + \alpha s_3)\Omega(x)]^{+} + [(1 + \alpha s_1)\Theta(x) + (1 + \alpha s_3)\Omega(x)]^{-}$$

$$= [\sigma_{yy} - \alpha\sigma_{xy}]^{+} + [\sigma_{yy} - \alpha\sigma_{xy}]^{-} \tag{3.12a}$$

and

$$[(1+\alpha s_1)\Theta(x)-(1+\alpha s_3)\Omega(x)]^+ - [(1+\alpha s_1)\Theta(x)-(1+\alpha s_3)\Omega(x)]^-$$
$$= [\sigma_{yy}-\alpha\sigma_{xy}]^+ - [\sigma_{yy}-\alpha\sigma_{xy}]^-. \quad (3.12b)$$

Since the right-hand-sides of Eqs. (3.12a) and (3.12b) are known, these equations are two non-homogeneous Hilbert problems for the two unknown functions $(1+\alpha s_1)\Theta(x)-(1+\alpha s_3)\Omega(x)$ and $(1+\alpha s_1)\Theta(x)+(1+\alpha s_3)\Omega(x)$. The solution to this problem is obtained from Eqs. (3.2a); see also, e.g. Muskhelishvili (1953).

3.1. A crack in an anisotropic plate loaded on a part of its upper face

Consider a straight crack in an infinitely extended plane. Some part of the upper surface of this crack is subjected to normal and shear tractions as shown in Fig. 2; note that the loads on the crack surfaces are not self-equilibrating. We seek to obtain the stress functions for this problem based on the Hilbert problem described above. The boundary conditions are as follows:

$$\sigma_{yy}^- = \sigma_{xy}^- = 0 \quad \text{for } -a < x < a, \quad (3.1.1a)$$

$$\sigma_{yy}^+ = -p \quad \text{and} \quad \sigma_{xy}^+ = -q \quad \text{for} \quad b < x < c, \quad (3.1.1b)$$

$$\sigma_{yy}^+ = \sigma_{xy}^+ = 0 \quad \text{for } -a < x < b \quad \text{and} \quad c < x < a. \quad (3.1.1c)$$

Use these boundary conditions in Eqs. (3.12a) and (3.12b) to arrive at

$$[(1+\alpha s_1)\Theta(x)+(1+\alpha s_3)\Omega(x)]^+ + [(1+\alpha s_1)\Theta(x)+(1+\alpha s_3)\Omega(x)]^- = -p+\alpha q \quad (3.1.2a)$$

and

$$[(1+\alpha s_1)\Theta(x)-(1+\alpha s_3)\Omega(x)]^+ - [(1+\alpha s_1)\Theta(x)-(1+\alpha s_3)\Omega(x)]^- = -p+\alpha q, \quad (3.1.2b)$$

for $b < x < c$ (otherwise zero). Considering Eqs. (3.1), (3.2a), the solution of the above boundary-value problem is

$$(1+\alpha s_1)\Theta(z)+(1+\alpha s_3)\Omega(z) = -\frac{p-\alpha q}{2\pi i}\frac{1}{\sqrt{z^2-a^2}}\left\{\int_b^c \frac{\sqrt{t^2-a^2}}{t-z}\,\mathrm{d}t + R(z)\right\} \quad (3.1.3a)$$

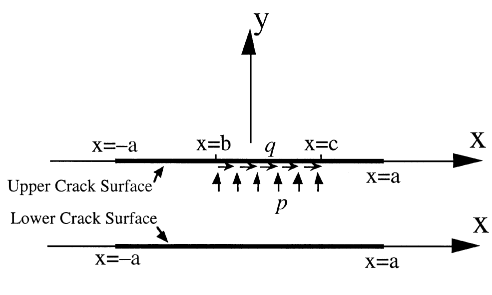

Fig. 2. An unbalanced loading condition; the upper crack face is subjected to the normal and tangential tractions, p and q, and the lower crack face is traction-free.

and

$$(1+\alpha s_1)\Theta(z)-(1+\alpha s_3)\Omega(z)=-\frac{p-\alpha q}{2\pi i}\int_b^c\frac{1}{t-z}\,\mathrm{d}t, \tag{3.1.3b}$$

where, according to Eqs. (3.2c), $R(z)=P_0$ (constant); this assures that the stresses decay to zero at infinity. Eq. (3.1.3b) comes directly from the Plemelj formula. Note that we must choose a proper branch of $\sqrt{z^2-a^2}$ according to the physical consideration of a line discontinuity problem. The two definite integrals in Eqs. (3.1.3a) and (3.1.3b) are obtained as follows:

$$\begin{aligned}\int_b^c\frac{\sqrt{t^2-a^2}}{t-z}\,\mathrm{d}t&=\int_b^c\frac{t}{\sqrt{t^2-a^2}}\,\mathrm{d}t+z\int_b^c\frac{1}{\sqrt{t^2-a^2}}\,\mathrm{d}t+(z^2-a^2)\int_b^c\frac{1}{(t-z)\sqrt{t^2-a^2}}\,\mathrm{d}t\\&=i\sqrt{a^2-c^2}-i\sqrt{a^2-b^2}-iz\left(\sin^{-1}\frac{c}{a}-\sin^{-1}\frac{b}{a}\right)\\&\quad+i\sqrt{z^2-a^2}\left\{\tan^{-1}\frac{cz-a^2}{\sqrt{a^2-c^2}\sqrt{z^2-a^2}}-\tan^{-1}\frac{bz-a^2}{\sqrt{a^2-b^2}\sqrt{z^2-a^2}}\right\}\end{aligned} \tag{3.1.4a}$$

and

$$\int_b^c\frac{1}{t-z}\,\mathrm{d}t=\log\frac{c-z}{b-z} \tag{3.1.4b}$$

Now, use Eqs. (3.1.4a) and (3.1.4b) to solve Eqs. (3.1.3a) and (3.1.3b) for the two unknown stress functions $\Theta(z)$ and $\Omega(z)$,

$$\Theta(z)=\frac{-(p-\alpha q)}{4\pi}\frac{1}{1+\alpha s_1}\left\{H(a,b,c,z)-i\log\frac{c-z}{b-z}-\frac{iP_0}{\sqrt{z^2-a^2}}\right\} \tag{3.1.5a}$$

and

$$\Omega(z)=\frac{-(p-\alpha q)}{4\pi}\frac{1}{1+\alpha s_3}\left\{H(a,b,c,z)+i\log\frac{c-z}{b-z}-\frac{iP_0}{\sqrt{z^2-a^2}}\right\}, \tag{3.1.5b}$$

where

$$\begin{aligned}H(a,b,c,z)&=\frac{\sqrt{a^2-c^2}-\sqrt{a^2-b^2}}{\sqrt{z^2-a^2}}-\frac{z}{\sqrt{z^2-a^2}}\left(\sin^{-1}\frac{c}{a}-\sin^{-1}\frac{b}{a}\right)\\&+\tan^{-1}\frac{cz-a^2}{\sqrt{a^2-c^2}\sqrt{z^2-a^2}}-\tan^{-1}\frac{bz-a^2}{\sqrt{a^2-b^2}\sqrt{z^2-a^2}}.\end{aligned} \tag{3.1.5c}$$

It still remains to determine the unknown constant P_0. First, consider the behavior of the functions Ω and Θ of Eqs. (3.1.5a) and (3.1.5b) at infinity. For this, recall that, for $|z|\to\infty$,

$$\frac{1}{\sqrt{z^2-a^2}}=O\left(\frac{1}{z}\right),\quad\frac{z}{\sqrt{z^2-a^2}}=1+O\left(\frac{1}{z^2}\right),\quad\log\frac{c-z}{b-z}=-\frac{c-b}{z}+O\left(\frac{1}{z^2}\right) \tag{3.1.6a}$$

and

$$\tan^{-1}\frac{cz-a^2}{\sqrt{a^2-c^2}\sqrt{z^2-a^2}}=\tan^{-1}\frac{c}{\sqrt{a^2-c^2}}-\frac{\sqrt{a^2-c^2}}{z}+O\left(\frac{1}{z^2}\right). \tag{3.1.6b}$$

Now, using Eqs. (3.1.6a,b), in Eq. (3.1.5c), we obtain $H(a, b, c, z)=O(1/z^2)$. Hence,

$$\Theta(z)=\frac{A}{z}+O\left(\frac{1}{z^2}\right),\quad \text{with}\quad A=\frac{p-\alpha q}{4\pi i}\frac{1}{1+\alpha s_1}(c-b-P_0) \tag{3.1.7a}$$

and

$$\Omega(z)=\frac{B}{z}+O\left(\frac{1}{z^2}\right),\quad \text{with}\quad B=-\frac{p-\alpha q}{4\pi i}\frac{1}{1+\alpha s_3}(c-b+P_0). \tag{3.1.7b}$$

Integration of the above yields

$$\theta(z)=A\log z+O\left(\frac{1}{z}\right)\quad \text{and}\quad \omega(z)=B\log z+O\left(\frac{1}{z}\right). \tag{3.1.8}$$

Based on Eqs. (3.1.8) and (3.10b), we rewrite Eq. (3.4b) as

$$\begin{aligned}u_x+\alpha u_y=&(p_1+\alpha q_1)A\log(z_1)+(p_3+\alpha q_3)B\log(z_3)+\frac{1}{2}(p_1+\alpha q_1)(A-\bar{B})(\log z_2-\log z_1)\\&-\frac{1}{2}(p_3+\alpha q_3)(\bar{A}-B)(\log z_4-\log z_3)+O\left(\frac{1}{z}\right).\end{aligned} \tag{3.1.9}$$

This displacement field has to be single-valued; see Eqs. (1.86) and (1.87) of Savin (1961) for a general discussion of this issue based on the classical expressions of the stress functions Φ and Ψ. This condition leads to

$$(p_1+\alpha q_1)A-(p_3+\alpha q_3)B=0. \tag{3.1.10}$$

Finally, substitute for A and B from Eqs. (3.1.7a) and (3.1.7b) in Eq. (3.1.10) to obtain P_0

$$P_0=(c-b)C_0,\quad C_0=\frac{(p_1+\alpha q_1)(1+\alpha s_3)+(p_3+\alpha q_3)(1+\alpha s_1)}{(p_1+\alpha q_1)(1+\alpha s_3)-(p_3+\alpha q_3)(1+\alpha s_1)}. \tag{3.1.11}$$

Appendix A provides an explicit expression for C_0 in terms of the material constants.

The original stress functions are now calculated via Eqs. (3.10a) and (3.10b) as

$$\begin{aligned}\Phi(z)=&\frac{1}{4\pi}\frac{ps_2+q}{s_1-s_2}\left\{H(a,b,c,\right.\\&\left.z)-i\log\frac{c-z}{b-z}\right\}-\frac{c-b}{8\pi i}\left\{\frac{(p-\alpha q)C_0}{1+\alpha s_1}-\frac{(p-\bar{\alpha}q)\bar{C}_0}{1+\bar{\alpha}s_1}\right\}\frac{1}{\sqrt{z^2-a^2}}\end{aligned} \tag{3.1.12a}$$

and

$$\Psi(z) = \frac{1}{4\pi}\frac{ps_1 + q}{s_2 - s_1}\left\{H(a, b, c, z) - i\log\frac{c-z}{b-z}\right\} - \frac{c-b}{8\pi i}\left\{\frac{(p-\alpha q)C_0}{1+\alpha s_2} - \frac{(p-\bar{\alpha}q)\bar{C}_0}{1+\bar{\alpha}s_2}\right\}\frac{1}{\sqrt{z^2-a^2}}. \tag{3.1.12b}$$

Of primary interest are the stress intensity factors (SIFs). Using the original definitions of SIFs for the right crack-tip, from Eq. (3.4a), on the real axis, the SIFs are

$$K_{\mathrm{I}} - \alpha K_{\mathrm{II}} = \lim_{x\to a^+} \sqrt{2\pi(x-a)}(\sigma_{yy} - \alpha\sigma_{xy})$$
$$= \lim_{x\to a^+} \sqrt{2\pi(x-a)}[(1+\alpha s_1)\Theta(x) + (1+\alpha s_3)\Omega(x)]. \tag{3.1.13}$$

Using Eqs. (3.1.5a), (3.1.5b), the stress intensity factors are obtained:

$$K_{\mathrm{I}} = -\frac{1}{2\sqrt{\pi a}}\left\{pM_-(a, b, c) + \frac{i(c-b)}{\alpha-\bar{\alpha}}\left[\bar{\alpha}(p-\alpha q)C_0 + \alpha(p-\bar{\alpha}q)\bar{C}_0\right]\right\} \tag{3.1.14a}$$

and

$$K_{\mathrm{II}} = -\frac{1}{2\sqrt{\pi a}}\left\{qM_-(a, b, c) + \frac{i(c-b)}{\alpha-\bar{\alpha}}\left[(p-\alpha q)C_0 + (p-\bar{\alpha}q)\bar{C}_0\right]\right\}, \tag{3.1.14b}$$

where

$$M_{\pm} = \sqrt{a^2-c^2} - \sqrt{a^2-b^2} \pm a\left(\sin^{-1}\frac{c}{a} - \sin^{-1}\frac{b}{a}\right). \tag{3.1.14c}$$

Similarly, the SIFs at the left crack-tip ($x=-a$) are given by

$$K_{\mathrm{I}} = \frac{1}{2\sqrt{\pi a}}\left\{pM_+(a, b, c) + \frac{i(c-b)}{\alpha-\bar{\alpha}}\left[\bar{\alpha}(p-\alpha q)C_0 + \alpha(p-\bar{\alpha}q)\bar{C}_0\right]\right\} \tag{3.1.15a}$$

and

$$K_{\mathrm{II}} = \frac{1}{2\sqrt{\pi a}}\left\{qM_+(a, b, c) + \frac{i(c-b)}{\alpha-\bar{\alpha}}\left[(p-\alpha q)C_0 + (p-\bar{\alpha}q)\bar{C}_0\right]\right\}. \tag{3.1.15b}$$

As is seen, the derivation of stress functions and the SIFs are quite simple by the present method as compared with the mapping method (see, e.g., Savin, 1961); in Section 4.4, the case of self-equilibrating tractions applied on the upper and lower crack faces is presented.

3.2. A crack in an anisotropic plate loaded on its entire upper face

This is a special case of the problem considered in Section 3.1, i.e. when the upper crack surface is loaded, while the lower one is traction free. Thus, the boundary conditions are:

$$\sigma_{yy}^+ = -p, \quad \sigma_{xy}^+ = -q \quad \text{for } -a < x < a \tag{3.2.1a}$$

and

$$\sigma_{yy}^{-} = \sigma_{xy}^{-} = 0 \quad \text{for} \; -a < x < a. \tag{3.2.1b}$$

In order to obtain the stress functions and the SIFs for this case, all the formulae of Section (3.1) are re-evaluated for the limiting case of $b \rightarrow -a$ and $c \rightarrow a$. The results are summarized below:

$$\Theta(z) = -\frac{p - \alpha q}{4\pi} \frac{1}{1 + \alpha s_1} \left\{ H(a, z) - i \log\frac{z - a}{z + a} - \frac{2aiC_0}{\sqrt{z^2 - a^2}} \right\}, \tag{3.2.2a}$$

$$\Omega(z) = -\frac{p - \alpha q}{4\pi} \frac{1}{1 + \alpha s_3} \left\{ H(a, z) + i \log\frac{z - a}{z + a} - \frac{2aiC_0}{\sqrt{z^2 - a^2}} \right\}, \tag{3.2.2b}$$

$$\Phi(z) = \frac{1}{4\pi} \frac{ps_2 + q}{s_1 - s_2} \left\{ H(a, z) - i \log\frac{z - a}{z + a} \right\} - \frac{a}{4\pi i} \left\{ \frac{(p - \alpha q)C_0}{1 + \alpha s_1} - \frac{(p - \bar{\alpha} q)\bar{C}_0}{1 + \bar{\alpha} s_1} \right\} \frac{1}{\sqrt{z^2 - a^2}} \tag{3.2.3a}$$

and

$$\Psi(z) = \frac{1}{4\pi} \frac{ps_1 + q}{s_2 - s_1} \left\{ H(a, z) - i \log\frac{z - a}{z + a} \right\} - \frac{a}{4\pi i} \left\{ \frac{(p - \alpha q)C_0}{1 + \alpha s_2} - \frac{(p - \bar{\alpha} q)\bar{C}_0}{1 + \bar{\alpha} s_2} \right\} \frac{1}{\sqrt{z^2 - a^2}}, \tag{3.2.3b}$$

where $H(a, z) = \pi(1 - \frac{z}{\sqrt{z^2 - a^2}})$ and the SIFs are

$$K_{\mathrm{I}} = \frac{1}{2\sqrt{\pi a}} \left\{ p\pi a - \frac{2ia}{\alpha - \bar{\alpha}} \left[\bar{\alpha}(p - \alpha q)C_0 + \alpha(p - \bar{\alpha} q)\bar{C}_0 \right] \right\}, \tag{3.2.4a}$$

$$K_{\mathrm{II}} = \frac{1}{2\sqrt{\pi a}} \left\{ q\pi a - \frac{2ia}{\alpha - \bar{\alpha}} \left[(p - \alpha q)C_0 + (p - \bar{\alpha} q)\bar{C}_0 \right] \right\} \tag{3.2.4b}$$

and

$$K_{\mathrm{I}} = \frac{1}{2\sqrt{\pi a}} \left\{ p\pi a + \frac{2ia}{\alpha - \bar{\alpha}} \left[\bar{\alpha}(p - \alpha q)C_0 + \alpha(p - \bar{\alpha} q)\bar{C}_0 \right] \right\}, \tag{3.2.5a}$$

$$K_{\mathrm{II}} = \frac{1}{2\sqrt{\pi a}} \left\{ q\pi a + \frac{2ia}{\alpha - \bar{\alpha}} \left[(p - \alpha q)C_0 + (p - \bar{\alpha} q)\bar{C}_0 \right] \right\} \tag{3.2.5b}$$

at $x = a$ and $x = -a$, respectively.

To obtain the SIFs for a crack loaded on its two faces by equal and opposite forces (self-equilibrating loads), assume that the same uniform loading that acts on the upper face acts only on the lower face. For this case, the SIFs are the same as Eqs. (3.2.4a), (3.2.4b), (3.2.5a) and (3.2.5b), except for a sign change for the terms involving C_0; this sign change ensures the single-valuedness of the displacements. Superposing the SIFs of the lower-loading case with the upper-loading case, yields the familiar expressions corresponding to the self-equilibrating uniform loading, i.e.,

$$K_{\mathrm{I}} = p\sqrt{\pi a} \quad \text{and} \quad K_{\mathrm{II}} = q\sqrt{\pi a} \tag{3.2.6}$$

for both ends ($x = -a$, and $x = a$).

Thus, as stated by Sih et al. (1965) and Barnett and Asaro (1972), for the case of self-balanced loading on the crack line, the SIFs are *independent* of the anisotropy of the medium.

3.3. A crack in an anisotropic plate loaded by a point load at an arbitrary point located on its upper face

Consider now, a case where a normal and a tangential point force, $-P$ and $-Q$, are applied at an arbitrary point $x=h$, $-a<h<a$, located on the upper surface of the crack; see Fig. 2. This is a special case of the one considered in Section (3.1) for which $b\to h^-$ and $c\to h^+$. The boundary conditions and the representation of the concentrated loads are as follows:

$$\sigma_{yy}^- = \sigma_{xy}^- = 0 \quad \text{for } -a<x<a, \tag{3.3.1a}$$

$$\sigma_{yy}^+ = -p, \quad \sigma_{xy}^+ = -q \quad \text{for } h-\varepsilon \le x \le h+\varepsilon, \tag{3.3.1b}$$

$$\lim_{\varepsilon\to 0}[2\varepsilon p] = P, \quad \lim_{\varepsilon\to 0}[2\varepsilon q] = Q. \tag{3.3.1c}$$

In order to obtain the stress functions and the SIFs for this case, all the formulae of Section 3.1 are modified for the limiting case of $b\to h^-$ and $c\to h^+$, according to Eqs. (3.3.1b) and (3.3.1c). For $O(\varepsilon^2)=0$, this limiting procedure changes functions H and M to

$$H(a,b,c,z)\pm i\log\frac{c-z}{b-z} \Longrightarrow \frac{2\varepsilon i}{z-h}\left(i\frac{\sqrt{a^2-h^2}}{\sqrt{z^2-a^2}}\mp 1\right), \quad M_{\pm}(a,b,c)\Longrightarrow \pm 2\varepsilon\frac{\sqrt{a\mp h}}{a\pm h}. \tag{3.3.2}$$

Thus, the stress functions of Section (3.1) become

$$\Theta(z) = \frac{-(P-\alpha Q)}{4\pi}\frac{i}{1+\alpha s_1}\left\{\frac{1}{z-h}\left(i\frac{\sqrt{a^2-h^2}}{\sqrt{z^2-a^2}}+1\right)-\frac{C_0}{\sqrt{z^2-a^2}}\right\}, \tag{3.3.3a}$$

$$\Omega(z) = \frac{-(P-\alpha Q)}{4\pi}\frac{i}{1+\alpha s_3}\left\{\frac{1}{z-h}\left(i\frac{\sqrt{a^2-h^2}}{\sqrt{z^2-a^2}}-1\right)-\frac{C_0}{\sqrt{z^2-a^2}}\right\}, \tag{3.3.3b}$$

$$\Phi(z) = \frac{i}{4\pi}\frac{Ps_2+Q}{s_1-s_2}\frac{1}{z-h}\left(i\frac{\sqrt{a^2-h^2}}{\sqrt{z^2-a^2}}+1\right)-\frac{1}{8\pi i}\left\{\frac{P-\alpha Q}{1+\alpha s_1}C_0-\frac{P-\bar{\alpha}Q}{1+\bar{\alpha}s_1}\bar{C}_0\right\}\frac{1}{\sqrt{z^2-a^2}}, \tag{3.3.4a}$$

$$\Psi(z) = \frac{i}{4\pi}\frac{Ps_1+Q}{s_2-s_1}\frac{1}{z-h}\left(i\frac{\sqrt{a^2-h^2}}{\sqrt{z^2-a^2}}+1\right)-\frac{1}{8\pi i}\left\{\frac{P-\alpha Q}{1+\alpha s_2}C_0-\frac{P-\bar{\alpha}Q}{1+\bar{\alpha}s_2}\bar{C}_0\right\}\frac{1}{\sqrt{z^2-a^2}} \tag{3.3.4b}$$

and the SIFs change to

$$K_{\mathrm{I}} = \frac{1}{2\sqrt{\pi a}}\left\{P\sqrt{\frac{a+h}{a-h}}-\frac{i}{\alpha-\bar{\alpha}}\left[\bar{\alpha}(P-\alpha Q)C_0+\alpha(P-\bar{\alpha}Q)\bar{C}_0\right]\right\}, \tag{3.3.5a}$$

$$K_{\mathrm{II}} = \frac{1}{2\sqrt{\pi a}}\left\{Q\sqrt{\frac{a+h}{a-h}} - \frac{i}{\alpha-\bar{\alpha}}\left[(P-\alpha Q)C_0 + (P-\bar{\alpha}Q)\bar{C}_0\right]\right\} \tag{3.3.5b}$$

at $x=a$, and

$$K_{\mathrm{I}} = \frac{1}{2\sqrt{\pi a}}\left\{P\sqrt{\frac{a-h}{a+h}} + \frac{i}{\alpha-\bar{\alpha}}\left[\bar{\alpha}(P-\alpha Q)C_0 + \alpha(P-\bar{\alpha}Q)\bar{C}_0\right]\right\}, \tag{3.3.5c}$$

$$K_{\mathrm{II}} = \frac{1}{2\sqrt{\pi a}}\left\{Q\sqrt{\frac{a-h}{a+h}} + \frac{i}{\alpha-\bar{\alpha}}\left[(P-\alpha Q)C_0 + (P-\bar{\alpha}Q)\bar{C}_0\right]\right\} \tag{3.3.5d}$$

at $x=-a$.

Next, let us compare Eqs. (3.3.5a–d) with the analogous results in the literature. Considering the form of the parameters α and C_0, (Eqs. (3.8) and (3.1.11)), it can be shown that the SIFs in Eqs. (3.3.5a) and (3.3.5b), which are for a general anisotropic case, are *exactly* the same as Eq. (38) of Sih et al. (1965) derived by the mapping method (note that they do not consider the tangential force Q). For the orthotropic cases reported in Appendix A, Eqs. (3.3.5a) and (3.3.5b) reduce to

$$K_{\mathrm{I}} = \frac{1}{2\sqrt{\pi a}}\left\{P\sqrt{\frac{a+h}{a-h}} + QC_{\mathrm{I}}\right\} \quad \text{and} \quad K_{\mathrm{II}} = \frac{1}{2\sqrt{\pi a}}\left\{Q\sqrt{\frac{a+h}{a-h}} + PC_{\mathrm{II}}\right\}, \tag{3.3.6a}$$

where

$$C_{\mathrm{I}} = \frac{1}{2\beta_0}\left[\frac{C_{12}}{C_{11}}\frac{1}{\alpha_0^2+\beta_0^2} + 1\right] \quad \text{and} \quad C_{\mathrm{II}} = -\frac{1}{2\beta_0}\left[\frac{C_{12}}{C_{11}} + \alpha_0^2 + \beta_0^2\right], \tag{3.3.6b}$$

or

$$C_{\mathrm{I}} = \frac{1}{\beta_1+\beta_2}\left[\frac{C_{12}}{C_{11}}\frac{1}{\beta_1\beta_2} + 1\right] \quad \text{and} \quad C_{\mathrm{II}} = -\frac{1}{\beta_1+\beta_2}\left[\frac{C_{12}}{C_{11}} + \beta_1\beta_2\right] \tag{3.3.6c}$$

for $s_1=\alpha_0+i\beta_0$ and $s_2=-\alpha_0+i\beta_0$ or $s_1=i\beta_1$ and $s_2=i\beta_2$, respectively. Note that Eq. (3.3.6a) does not agree with Eq. (40) of Sih et al. (1965) because their Eq. (39) contains a minor error, i.e. the last term of Eq. (39) must change from '+1' to '+2'. It is noteworthy that, according to Eq. (3.3.6a), in an orthotropic plane, if only a normal (tangential) force acts at any point on the crack face, it creates only Mode-I SIF (Mode-II SIF) which is independent of the material properties. Moreover, using the explicit form of α and C_0 given in Appendix A, Eqs. (3.3.5a–d) reduce to the ones derived by Sih and Liebowitz (1968) for the isotropic case, which are

$$K_{\mathrm{I}} = \frac{1}{2\sqrt{\pi a}}\left\{P\sqrt{\frac{a+h}{a-h}} + Q\frac{\kappa-1}{\kappa+1}\right\} \quad \text{and} \quad K_{\mathrm{II}} = \frac{1}{2\sqrt{\pi a}}\left\{Q\sqrt{\frac{a+h}{a-h}} - P\frac{\kappa-1}{\kappa+1}\right\} \quad \text{at } x=a; \tag{3.3.7a}$$

$$K_{\mathrm{I}} = \frac{1}{2\sqrt{\pi a}}\left\{P\sqrt{\frac{a-h}{a+h}} - Q\frac{\kappa-1}{\kappa+1}\right\} \quad \text{and} \quad K_{\mathrm{II}} = \frac{1}{2\sqrt{\pi a}}\left\{Q\sqrt{\frac{a-h}{a+h}} + P\frac{\kappa-1}{\kappa+1}\right\} \quad \text{at } x=-a, \tag{3.3.7b}$$

where the forces P and Q are assumed to be applied at point $x=h$ located on the upper crack surface, and $\kappa=3-4\nu$ and $\kappa=(3-\nu)/(1+\nu)$ for plane-strain and plane-stress conditions, respectively.

Similar to the discussion given in Section 3.2, if the concentrated forces are applied only on the lower crack surfaces, then the SIFs of Eqs. (3.3.5a–d) remain the same, except for a sign change for all the terms containing C_0. Considering this fact, the SIFs for the case where both crack surfaces are loaded by concentrated forces of equal magnitude but opposite signs, (balanced forces) are

$$K_{\mathrm{I}} = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a+h}{a-h}} \quad \text{and} \quad K_{\mathrm{II}} = \frac{Q}{\sqrt{\pi a}}\sqrt{\frac{a+h}{a-h}} \quad \text{at } x = a; \tag{3.3.8a}$$

$$K_{\mathrm{I}} = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a-h}{a+h}} \quad \text{and} \quad K_{\mathrm{II}} = \frac{Q}{\sqrt{\pi a}}\sqrt{\frac{a-h}{a+h}} \quad \text{at } x = -a. \tag{3.3.8b}$$

Eqs. (3.3.5a–d), Eqs. (3.3.6a), Eqs. (3.3.7a,b), and (3.3.8a,b) are the corresponding *Green functions*. They are used to calculate the SIFs at the crack tips due to the unbalanced (Eqs. (3.3.5a–d), Eqs. (3.3.6a), (3.3.7a,b)) and/or balanced (Eqs. (3.3.8a,b)) forces on the crack line; see also Section 4.4. As an example, consider distributed normal and tangential tractions, $p(x)$ and $q(x)$, applied on the upper surface (between points u and v) of the crack. The resulting SIFs at the right tip are

$$K_{\mathrm{I}} = \frac{1}{2\sqrt{\pi a}}\int_u^v \left\{ p(\zeta)\sqrt{\frac{a+\zeta}{a-\zeta}} - \frac{i}{\alpha-\bar{\alpha}}\left[\bar{\alpha}(p(\zeta)-\alpha q(\zeta))C_0 + \alpha(p(\zeta)-\bar{\alpha}q(\zeta))\bar{C}_0\right]\right\} \mathrm{d}\zeta \tag{3.3.9a}$$

and

$$K_{\mathrm{II}} = \frac{1}{2\sqrt{\pi a}}\int_u^v \left\{ q(\zeta)\sqrt{\frac{a+\zeta}{a-\zeta}} - \frac{i}{\alpha-\bar{\alpha}}\left[(p(\zeta)-\alpha q(\zeta))C_0 + (p(\zeta)-\bar{\alpha}q(\zeta))\bar{C}_0\right]\right\} \mathrm{d}\zeta. \tag{3.3.9b}$$

3.4. A half-plane loaded on its free surface (the plane $y = 0$)

As mentioned earlier, one advantage of the Hilbert formulation is that it can be used to solve the problem of a crack loaded only on one of its faces. The results may also be used to generate solutions for a half-plane subjected to various surface tractions. For example, taking the limit of Eqs. (3.1.12a), (3.1.12b), (3.2.3a), (3.2.3b), (3.3.4a) and (3.3.4b), as 'a' goes to infinity, yields the solution for a half-plane ($y \geq 0$) loaded: on a finite part of its surface ($b \leq x \leq c$ and $y = 0$), on its entire surface ($-\infty \leq x \leq \infty$ and $y = 0$), or at a single point on its surface ($x=h$ and $y = 0$), respectively. For this, the following results are helpful:

$$\lim_{a\to\infty}\left[\tan^{-1}\frac{cz-a^2}{\sqrt{a^2-c^2}\sqrt{z^2-a^2}} - \tan^{-1}\frac{bz-a^2}{\sqrt{a^2-b^2}\sqrt{z^2-a^2}}\right] = i\log\frac{c-z}{b-z} \quad \text{for } y<0 \tag{3.4.1a}$$

and

$$\lim_{a\to\infty}\left[\tan^{-1}\frac{cz-a^2}{\sqrt{a^2-c^2}\sqrt{z^2-a^2}} - \tan^{-1}\frac{bz-a^2}{\sqrt{a^2-b^2}\sqrt{z^2-a^2}}\right] = -i\log\frac{c-z}{b-z} \quad \text{for } y>0 \tag{3.4.1b}$$

Using these, the stress functions (Eqs. (3.1.12a), (3.1.12b), (3.2.3a), (3.2.3b), (3.3.4a) and (3.3.4b)) are reduced to

$$\Phi(z) = -i\frac{ps_2 + q}{2\pi(s_1 - s_2)} \log\frac{c - z}{b - z}, \quad \Psi(z) = -i\frac{ps_1 + q}{2\pi(s_2 - s_1)} \log\frac{c - z}{b - z}, \tag{3.4.2}$$

$$\Phi = \frac{ps_2 + q}{2(s_1 - s_2)}, \quad \Psi = \frac{ps_1 + q}{2(s_2 - s_1)}, \tag{3.4.3}$$

$$\Phi(z) = \frac{i}{2\pi}\frac{Ps_2 + Q}{s_1 - s_2}\frac{1}{z - h} \quad \text{and} \quad \Psi(z) = \frac{i}{2\pi}\frac{Ps_1 + Q}{s_2 - s_1}\frac{1}{z - h} \tag{3.4.4}$$

for the upper plane ($y > 0$, which is loaded). For the lower plane ($y < 0$, which is load-free), these stress functions are obtained to be identically *zero*. Note that the above results can also be obtained from Eqs. (4.4.16a), (4.4.16b), (4.4.8a), (4.4.8b), (4.4.12a) and (4.4.12b) of Section 4.4. These results have been reported in the literature (e.g., for the anisotropic case: Green and Zerna, 1954; Savin, 1961; Lekhnitskii, 1956; Lekhnitskii, 1963; and for the isotropic case: Muskhelishvili, 1953; Timoshenko and Goodier, 1970) using different methods.

4. Method of continuously distributed dislocations (CDD)

In this section, a crack is modeled by a continuously distributed dislocations along its line. This method is well established and many isotropic and some anisotropic crack problems are solved using this method. Here, using this method, we present the solution of five anisotropic crack problems with balanced loads on their faces. While some of these results are new (e.g., Section 4.5), some are the extension of the existing results for isotropic problems to the anisotropic case (e.g., Sections 4.2 and 4.3.1). The analysis in Section 4.4 demonstrates the contrast between the Hilbert and CDD methods, showing how one method can produce the solution with greater ease than the other.

4.1. Green's functions for an open crack with an edge dislocation

Consider an infinitely extended anisotropic body containing a straight crack with length $2a$. The farfield stresses, σ_{xx}^{∞}, σ_{yy}^{∞} and σ_{xy}^{∞}, are such that the crack surfaces are *not* in contact (the conditions under which the crack remains open are obtained at the end of this section). In addition to this, a single edge dislocation $b^0 = (b_x^0, b_y^0)$, is located at an arbitrary point $z^0 = (x^0, y^0)$ (CASE #1), or a pair of centrally-symmetric edge-dislocations, (b_x^0, b_y^0) and $(-b_x^0, -b_y^0)$, are located at (x^0, y^0) and $(-x^0, -y^0)$ (CASE #2); see Fig. 3. For the isotropic case, Lo (1978) formulated the Green function for this problem and used it to model kinks emanating from the tips of an existing crack. Obata et al. (1989) applied Lo's method to anisotropic media. Here, we present their formulation in detail and correct a minor error; see Eq. (4.1.12a). The free-surface conditions are

$$\sigma_{yy}(x, 0) = 0, \quad \sigma_{xy}(x, 0) = 0, \quad \textit{for} \ -a \leqslant x \leqslant a \tag{4.1.1}$$

where

$$\sigma_{yy}(x, 0) = \sigma_{yy}^{\mathrm{c}}(x, 0) + \sigma_{yy}^{\infty} + \sigma_{yy}^{\mathrm{D}}(x, 0; x^0, y^0) + \sigma_{yy}^{\mathrm{d}}(x, 0; -x^0, -y^0) \tag{4.1.2a}$$

and

$$\sigma_{xy}(x,0)=\sigma^{c}_{xy}(x,0)+\sigma^{\infty}_{xy}+\sigma^{D}_{xy}(x,0;\,x^0,\,y^0)+\sigma^{d}_{xy}(x,0;\,-x^0,\,-y^0). \qquad (4.1.2b)$$

In these expressions, the following notations are used:

1. The two terms with superscript 'c' represent the stresses caused by the presence of the distributed edge dislocations, b^{c}, along the crack. Since, on the crack line, $z_1=z_2=x$ and $z_1^0=z_2^0=t$, Eqs. (2.5b), (2.5c), (2.10a) and (2.10b)) give

$$\sigma^{c}_{yy}(x,0)=2\,\mathrm{Real}\left[\frac{1}{2\pi iC_{11}}\int_{-a}^{a}\frac{1}{x-t}\left\{\frac{s_1b^{c}_x(t)-b^{c}_y(t)}{\tilde{s}_1}+\frac{s_2b^{c}_x(t)-b^{c}_y(t)}{\tilde{s}_2}\right\}\mathrm{d}t\right] \qquad (4.1.3a)$$

and

$$\sigma^{c}_{xy}(x,0)=2\,\mathrm{Real}\left[\frac{-1}{2\pi iC_{11}}\int_{-a}^{a}\frac{1}{x-t}\left\{\frac{s_1\left(s_1b^{c}_x(t)-b^{c}_y(t)\right)}{\tilde{s}_1}+\frac{s_2\left(s_2b^{c}_x(t)-b^{c}_y(t)\right)}{\tilde{s}_2}\right\}\mathrm{d}t\right]. \qquad (4.1.3b)$$

2. The two terms with superscript 'D' denote the stresses at point $(x, 0)$ generated by the presence of a single edge-dislocation $b^0=(b_x^0,\,b_y^0)$ located at a generic point $z^0=(x^0,\,y^0)$. From Eqs. (2.5b), (2.5c), (2.10a) and (2.10b),

$$\sigma^{D}_{yy}(x,0;\,x^0,\,y^0)=2\,\mathrm{Real}\left[\frac{1}{2\pi iC_{11}}\left\{\frac{s_1b^0_x-b^0_y}{\tilde{s}_1}\frac{1}{x-z_1^0}+\frac{s_2b^0_x-b^0_y}{\tilde{s}_2}\frac{1}{x-z_2^0}\right\}\right] \qquad (4.1.4a)$$

and

$$\sigma^{D}_{xy}(x,0;\,x^0,\,y^0)=2\,\mathrm{Real}\left[\frac{-1}{2\pi iC_{11}}\left\{\frac{s_1\left(s_1b^0_x-b^0_y\right)}{\tilde{s}_1}\frac{1}{x-z_1^0}+\frac{s_2\left(s_2b^0_x-b^0_y\right)}{\tilde{s}_2}\frac{1}{x-z_2^0}\right\}\right]. \qquad (4.1.4b)$$

3. The two terms with superscript 'd' denote the stresses at point $(x, 0)$ generated by the presence of a single edge-dislocation $-b^0=(-b_x^0,\,-b_y^0)$ at point $-z^0=(-x^0,\,-y^0)$,

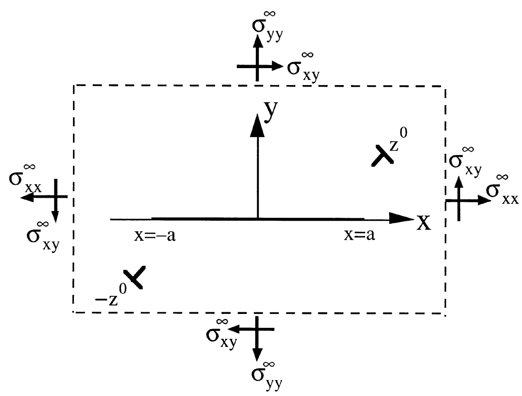

Fig. 3. A body containing an open crack and two centrally-symmetric edge-dislocations at z^0 and $-z^0$ under farfield stresses.

$$\sigma^{\rm d}_{yy}(x,0;-x^0,-y^0) = -2\,{\rm Real}\left[\frac{1}{2\pi i C_{11}}\left\{\frac{s_1 b^0_x - b^0_y}{\tilde{s}_1}\frac{1}{x+z^0_1} + \frac{s_2 b^0_x - b^0_y}{\tilde{s}_2}\frac{1}{x+z^0_2}\right\}\right] \quad (4.1.4c)$$

and

$$\sigma^{\rm d}_{xy}(x,0;\,-x^0,\,-y^0)$$
$$= -2\,{\rm Real}\left[\frac{-1}{2\pi i C_{11}}\left\{\frac{s_1\left(s_1 b^0_x - b^0_y\right)}{\tilde{s}_1}\frac{1}{x+z^0_1} + \frac{s_2\left(s_2 b^0_x - b^0_y\right)}{\tilde{s}_2}\frac{1}{x+z^0_2}\right\}\right]. \quad (4.1.4d)$$

Note that for CASE #2, terms with superscripts 'D' and 'd' must be retained in Eqs. (4.1.2a) and (4.1.2b), whereas the last term with superscript 'd' must be excluded for CASE #1.

Now, combine Eqs. (4.1.1), (4.1.2a), (4.1.2b), (4.1.3a), (4.1.3b), (4.1.4a–d) to arrive at the following integral equations:

$$2\,{\rm Real}\left[\frac{1}{2\pi i C_{11}}\int_{-a}^{a}\frac{1}{x-t}\left\{\frac{s_1 b^{\rm c}_x(t) - b^{\rm c}_y(t)}{\tilde{s}_1} + \frac{s_2 b^{\rm c}_x(t) - b^{\rm c}_y(t)}{\tilde{s}_2}\right\}{\rm d}t\right]$$
$$+\sigma^{\infty}_{yy} + \sigma^{\rm D}_{yy}(x,0;x^0,y^0) + \sigma^{\rm d}_{yy}(x,0;\,-x^0,\,-y^0) = 0 \quad (4.1.5a)$$

and

$$2\,{\rm Real}\left[\frac{-1}{2\pi i C_{11}}\int_{-a}^{a}\frac{1}{x-t}\left\{\frac{s_1\left(s_1 b^{\rm c}_x(t) - b^{\rm c}_y(t)\right)}{\tilde{s}_1} + \frac{s_2\left(s_2 b^{\rm c}_x(t) - b^{\rm c}_y(t)\right)}{\tilde{s}_2}\right\}{\rm d}t\right]$$
$$+\sigma^{\infty}_{xy} + \sigma^{\rm D}_{xy}(x,0;x^0,y^0) + \sigma^{\rm d}_{xy}(x,0;\,-x^0,\,-y^0) = 0. \quad (4.1.5b)$$

Using Eqs. (B1a) and (B1b), the solution to Eqs. (4.1.5a) and (4.1.5b) is

$$2\,{\rm Real}\left[\frac{1}{C}\left\{\frac{s_1 b^{\rm c}_x(t) - b^{\rm c}_y(t)}{\tilde{s}_1} + \frac{s_2 b^{\rm c}_x(t) - b^{\rm c}_y(t)}{\tilde{s}_2}\right\}\right] = \frac{\sigma^{\infty}_{yy}}{\pi}\frac{x}{\sqrt{a^2-x^2}} + \frac{1}{\pi^2\sqrt{a^2-x^2}}$$
$$2\,{\rm Real}\left[\frac{1}{C}\int_{-a}^{a}\frac{\sqrt{a^2-t^2}}{x-t}\left\{\frac{s_1 b^0_x - b^0_y}{\tilde{s}_1}\left(\frac{1}{t-z^0_1} - \frac{1}{t+z^0_1}\right) + \frac{s_2 b^0_x - b^0_y}{\tilde{s}_2}\left(\frac{1}{t-z^0_2} - \frac{1}{t+z^0_2}\right)\right\}{\rm d}t\right], \quad (4.1.6a)$$

and

$$2\,{\rm Real}\left[\frac{-1}{C}\left\{\frac{s_1\left(s_1 b^{\rm c}_x(t) - b^{\rm c}_y(t)\right)}{\tilde{s}_1} + \frac{s_2\left(s_2 b^{\rm c}_x(t) - b^{\rm c}_y(t)\right)}{\tilde{s}_2}\right\}\right] = \frac{\sigma^{\infty}_{xy}}{\pi}\frac{x}{\sqrt{a^2-x^2}} + \frac{1}{\pi^2\sqrt{a^2-x^2}}$$

and

$$2\,\mathrm{Real}\left[\frac{-1}{C}\int_{-a}^{a}\frac{\sqrt{a^2-t^2}}{x-t}\left\{\frac{s_1\left(s_1b_x^0-b_y^0\right)}{\tilde{s}_1}\left(\frac{1}{t-z_1^0}-\frac{1}{t+z_1^0}\right)\right.\right.$$
$$\left.\left.+\frac{s_2\left(s_2b_x^0-b_y^0\right)}{\tilde{s}_2}\left(\frac{1}{t-z_2^0}-\frac{1}{t+z_2^0}\right)\right\}\mathrm{d}t\right], \tag{4.1.6b}$$

where $C=2\pi iC_{11}$. Now, first we use Eq. (B2e) for the definite integrals in Eqs (4.1.6a,b), and then solve Eqs (4.1.6a,b) for the two unknowns b_x^c and b_y^c. The final results are:

$$b_x^c(x)=iC_{11}\left\{(s_3s_4-s_1s_2)\sigma_{yy}^\infty+(s_3+s_4-s_1-s_2)\sigma_{xy}^\infty\right\}\frac{x}{\sqrt{a^2-x^2}}+\frac{1}{2\pi\sqrt{a^2-x^2}}$$
$$\left\{\frac{\left(s_1b_x^0-b_y^0\right)}{s_1-s_2}X_1+\frac{\left(s_2b_x^0-b_y^0\right)}{s_2-s_1}X_2+\frac{\left(s_3b_x^0-b_y^0\right)}{s_3-s_4}X_3+\frac{\left(s_4b_x^0-b_y^0\right)}{s_4-s_3}X_4\right\} \tag{4.1.7a}$$

and

$$b_y^c(x)=iC_{11}\left\{\left[s_1s_3(s_4-s_2)+s_2s_4(s_3-s_1)\right]\sigma_{yy}^\infty+(s_3s_4-s_1s_2)\sigma_{xy}^\infty\right\}\frac{x}{\sqrt{a^2-x^2}}+\frac{1}{2\pi\sqrt{a^2-x^2}}$$
$$\left\{\frac{s_2\left(s_1b_x^0-b_y^0\right)}{s_1-s_2}X_1+\frac{s_1\left(s_2b_x^0-b_y^0\right)}{s_2-s_1}X_2+\frac{s_4\left(s_3b_x^0-b_y^0\right)}{s_3-s_4}X_3+\frac{s_3\left(s_4b_x^0-b_y^0\right)}{s_4-s_3}X_4\right\}, \tag{4.1.7b}$$

where X_j for CASE #1 and CASE #2 is given by

$$X_j=1+\frac{\sqrt{z_j^{0^2}-a^2}}{x-z_j^0} \tag{4.1.7c}$$

and

$$X_j=\frac{\sqrt{z_j^{0^2}-a^2}}{x-z_j^0}+\frac{\sqrt{z_j^{0^2}-a^2}}{x+z_j^0},\ (j=1,2,3,4), \tag{4.1.7d}$$

respectively. If dislocations of densities $b_x^c(x)$ and $b_y^c(x)$, given by Eqs. (4.1.7a) and (4.1.7b), are distributed along the crack line, then the crack surfaces will remain open (and thus, stress free); see the required restrictions on the applied loads given by Eq. (4.1.15). Note that the first terms in Eqs. (4.1.7a) and (4.1.7b) are due to the farfield loading and the second terms are due to the presence of the single edge dislocation.

Now, find the potential functions (equivalently, find the stresses) due to the presence of $b_x^c(x)$ and $b_y^c(x)$ along the crack line. Substitution of $b_x^c(x)$ and $b_y^c(x)$ into Eqs. (2.10a,b), for b_x^0 and b_y^0 gives the potential functions,

$$\Phi_M(z_1)\equiv\frac{1}{2\pi iC_{11}}\frac{1}{\tilde{s}_1}\int_{-a}^{a}\frac{s_1b_x^c(t)-b_y^c(t)}{z_1-t}\,\mathrm{d}t \tag{4.1.8a}$$

and

$$\Psi_M(z_2) \equiv \frac{1}{2\pi i C_{11}} \frac{1}{\tilde{s}_2} \int_{-a}^{a} \frac{s_2 b_x^c(t) - b_y^c(t)}{z_2 - t}\, dt. \tag{4.1.8b}$$

After substituting for $b_x^c(t)$ and $b_y^c(x)$ from Eqs. (4.1.7a,b) into Eqs. (4.1.8a) and (4.1.8b), definite integrals similar to Eqs. (B2a), (B2b) and (B2c) are encountered. Using these formulae and performing some rather cumbersome algebra, the following expressions for the potential functions Φ_M and Ψ_M are obtained:

$$\Phi_M(z_1; z^0) = \Phi^\infty(z_1) + \Phi^M(z_1; z^0) \quad \Psi_M(z_2; z^0) = \Psi^\infty(z_2) + \Psi^M(z_2; z^0), \tag{4.1.9}$$

where

$$\Phi^\infty(z_1) = \frac{s_2\sigma_{yy}^\infty + \sigma_{xy}^\infty}{2(s_1 - s_2)} \left\{ 1 - \frac{z_1}{\sqrt{z_1^2 - a^2}} \right\}, \quad \Psi^\infty(z_2) = \frac{s_1\sigma_{yy}^\infty + \sigma_{xy}^\infty}{2(s_2 - s_1)} \left\{ 1 - \frac{z_2}{\sqrt{z_2^2 - a^2}} \right\}, \tag{4.1.10}$$

$$\begin{aligned} \Phi^M(z_1; z^0) &= \frac{1}{4\pi i C_{11}} \frac{1}{s_1 - s_2} \\ &\times \left\{ \frac{s_1 b_x^0 - b_y^0}{(s_1 - s_3)(s_1 - s_4)} Y_{11} + \frac{s_3 b_x^0 - b_y^0}{(s_1 - s_3)(s_3 - s_4)} Y_{13} + \frac{s_4 b_x^0 - b_y^0}{(s_1 - s_4)(s_4 - s_3)} Y_{14} \right\} \end{aligned} \tag{4.1.11a}$$

and

$$\begin{aligned} \Psi^M(z_2; z^0) &= \frac{1}{4\pi i C_{11}} \frac{1}{s_2 - s_1} \\ &\times \left\{ \frac{s_2 b_x^0 - b_y^0}{(s_2 - s_3)(s_2 - s_4)} Y_{21} + \frac{s_3 b_x^0 - b_y^0}{(s_2 - s_3)(s_3 - s_4)} Y_{23} + \frac{s_4 b_x^0 - b_y^0}{(s_2 - s_4)(s_4 - s_3)} Y_{24} \right\}, \end{aligned} \tag{4.1.11b}$$

where

$$Y_{ij} = \frac{1}{\sqrt{z_i^2 - a^2}} + \frac{1}{z_i - z_j^0} \left\{ \frac{\sqrt{z_j^{0^2} - a^2}}{\sqrt{z_i^2 - a^2}} - 1 \right\}, \quad i = 1, 2 \text{ and } j = 1, 2, 3, 4, (\text{ for CASE \#1}), \tag{4.1.12a}$$

and

$$Y_{ij} = \frac{1}{z_i - z_j^0} \left\{ \frac{\sqrt{z_j^{0^2} - a^2}}{\sqrt{z_i^2 - a^2}} - 1 \right\} + \frac{1}{z_i + z_j^0} \left\{ \frac{\sqrt{z_j^{0^2} - a^2}}{\sqrt{z_i^2 - a^2}} + 1 \right\} \quad (\text{ for CASE \#2}). \tag{4.1.12b}$$

Note that, in contrast to Eqs. (2.10a) and (2.10b), functions Y_{ij} are not singular at $z_i = z_j^0$. This can be clarified by noting that

$$\frac{1}{z_i - z_j^0}\left\{\frac{\sqrt{z_j^{0^2} - a^2}}{\sqrt{z_i^2 - a^2}} - 1\right\} = \frac{-\left(z_i + z_j^0\right)}{\sqrt{z_i^2 - a^2}\left(\sqrt{z_i^2 - a^2} + \sqrt{z_j^{0^2} - a^2}\right)}. \tag{4.1.13}$$

The potential functions in Eqs. (4.1.10) are associated with the applied uniform loads; these are identical to the ones given by Savin (1961) who derived them by the method of *collapsing an ellipsoidal hole to a crack*. Moreover, the potential functions in Eqs. (4.1.11a) and (4.1.11b) correspond to the effects of a single edge dislocation. These potentials are the corresponding *Green's functions*.

The above results (Green's functions) can be used to solve various crack problems, such as:

1. a kink emanating from the tip of a pre-existing crack;
2. two centrally symmetric kinks emanating from the tips of a pre-existing crack;
3. two cracks with any size and any relative orientation with respect to each other (one of which can be considered as the pre-existing crack); and
4. three cracks with the second and third one having any size and orientation, but being centrally symmetric with respect to the first one (the pre-existing crack).

The main advantage of the above Green functions are that, for any of these four problems, *just one* line of discontinuity (kink or the crack) has to be modeled by the continuous distribution of the edge dislocations along the line, because the presence of the pre-existing crack is already built into the Green function.

The first problem of the previous paragraph is considered by Obata et al. (1989) and Azhdari and Nemat-Nasser (1996b). They used an equation similar to Eq. (4.1.12a), but omitting the first term. All their calculations are for the case of a *vanishingly small kink* emanated from the tip of a pre-existing crack. For this particular application, the effect of the first term vanishes as the kink length approaches zero. Therefore, *no error* is introduced by omitting the first term in Eq. (4.1.12a) in this particular case. It is interesting to note that, when two centrally-symmetric kinks (or, equivalently, two centrally-symmetric edge dislocations) are considered, then the first term in Eq. (4.1.12a) makes no contribution to the final result (see Eq. (4.1.12b)).

Let us, now, calculate the crack-opening displacement (COD) along the crack line. For this, we only use the first terms of Eqs. (4.1.7a) and (4.1.7b) in Eqs. (C4a) and (C4b), respectively, to arrive at

$$U_x(x) = 2C_{11}\left[(\alpha_1\beta_2 + \alpha_2\beta_1)\sigma_{yy}^{\infty} + (\beta_1 + \beta_2)\sigma_{xy}^{\infty}\right]\sqrt{a^2 - x^2} \tag{4.1.14a}$$

and

$$U_y(x) = 2C_{11}\left[\left(\beta_1\left(\alpha_2^2 + \beta_2^2\right) + \beta_2\left(\alpha_1^2 + \beta_1^2\right)\right)\sigma_{yy}^{\infty} + (\alpha_1\beta_2 + \alpha_2\beta_1)\sigma_{xy}^{\infty}\right]\sqrt{a^2 - x^2}. \tag{4.1.14b}$$

For an *open* crack, the y-component of the COD Eq. (4.1.14b) must be non-negative to ensure no material inter-penetration. Thus, for all the formulations of this section, it must be required that

$$\left[\left(\beta_1\left(\alpha_2^2 + \beta_2^2\right) + \beta_2\left(\alpha_1^2 + \beta_1^2\right)\right)\sigma_{yy}^{\infty} + (\alpha_1\beta_2 + \alpha_2\beta_1)\sigma_{xy}^{\infty}\right] > 0. \tag{4.1.15}$$

This completes the formulation of the problem of an open crack under farfield loads, and in the presence of a single or a pair of centrally-symmetric edge dislocations. Application of this formulation is demonstrated by Obata et al. (1989) and Azhdari and Nemat-Nasser (1996a, 1996b).

As a final comment for this section, we calculate the stress intensity factors (SIFs) at the crack tip, for the case when the open crack is subjected to only the farfield loads. Substitution of Eqs. (4.1.10) in Eqs. (2.5b) and (2.5c) gives the stresses, and the SIFs become (see Appendix D.1)

$$K_{\rm I} = \sigma_{yy}^{\infty}\sqrt{\pi a} \quad \text{and} \quad K_{\rm II} = \sigma_{xy}^{\infty}\sqrt{\pi a}. \tag{4.1.16}$$

Therefore, as stated by Sih et al. (1965) and Barnett and Asaro (1972), for an open crack in a medium with any degree of anisotropy, the SIFs are not dependent upon the material properties and they are identical to the SIFs of the isotropic case; compare this comment with the one given in Section 4.2.

4.2. Green's functions for a closed frictional crack with an edge dislocation

Consider a straight crack with length $2a$ in an infinitely extended anisotropic body. The farfield axial and lateral loads (σ_{11}^{∞} and σ_{22}^{∞}) are such that the crack surfaces are in contact; see Fig. 4. Therefore, the y-component of COD is zero (see Eq. (4.1.15)). The crack faces may slide against each other. In such a case, we assume a Coulomb-type frictional and cohesive contact with friction coefficient μ and cohesive stress τ_c, resisting the sliding. In addition, the body contains either a single edge dislocation (CASE #1) or a pair of centrally-symmetric edge dislocations (CASE #2); see Fig. 4. The formulation of this problem is somewhat similar to Section 4.1, and thus, we present just the key results; for the isotropic case, see the formulation by Horii and Nemat-Nasser (1985).

In order to create the frictional conditions on the crack surfaces, the following distribution of dislocations should exist along the crack line as in Eqs. (4.1.7a,b):

$$b_x^{\rm c}(x) = \frac{\sigma_{xy}^{\rm eq.}}{\pi Q}\frac{x}{\sqrt{a^2-x^2}} - \frac{1}{2\pi^2 iC_{11}Q}\frac{1}{\sqrt{a^2-x^2}}(W_1+W_2-W_3-W_4), \tag{4.2.1}$$

$$W_j = (s_j+\mu)\frac{s_j b_x^0 - b_y^0}{\tilde{s}_j}\left(1+\frac{\sqrt{z_j^{0\,2}-a^2}}{x-z_j^0}\right), \quad j=1,2,3,4 \quad (\text{ for CASE \#1}) \tag{4.2.2a}$$

and

$$W_j = (s_j+\mu)\frac{s_j b_x^0 - b_y^0}{\tilde{s}_j}\left(\frac{\sqrt{z_j^{0\,2}-a^2}}{x-z_j^0}+\frac{\sqrt{z_j^{0\,2}-a^2}}{x+z_j^0}\right), \quad j=1,2,3,4 \quad (\text{ for CASE \#2}) \tag{4.2.2b}$$

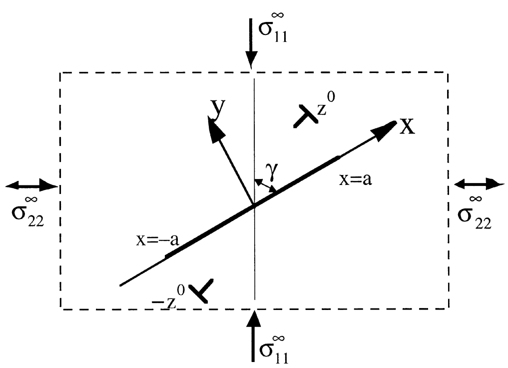

Fig. 4. A body containing a frictional crack and two centrally-symmetric edge-dislocations at z^0 and $-z^0$ subjected to farfield axial compressive and lateral tensile or compressive loads.

where

$$Q = 2\,\mathrm{Real}\left[\frac{-1}{2\pi i C_{11}}\left\{\left(\frac{s_1^2}{\tilde{s}_1}+\frac{s_2^2}{\tilde{s}_2}\right)+\mu\left(\frac{s_1}{\tilde{s}_1}+\frac{s_2}{\tilde{s}_2}\right)\right\}\right] = \frac{-1}{\pi C_{11}}\left(\frac{\Delta_3+\mu\Delta_2}{\Delta_0}\right) \tag{4.2.3a}$$

and

$$\sigma_{xy}^{\mathrm{eq.}} = \sigma_{xy}^{\infty} + \tau_{\mathrm{c}} - \mu\sigma_{yy}^{\infty}, \tag{4.2.3b}$$

where parameters Δ_j are defined in Appendix A; the term $\sigma_{xy}^{\mathrm{eq.}}$ is the *equivalent* shear stress on the crack line, $\sigma_{yy}^{\infty} = \sigma_{22}^{\infty}\cos^2\gamma + \sigma_{11}^{\infty}\sin^2\gamma$ and $\sigma_{xy}^{\infty} = (\sigma_{11}^{\infty} - \sigma_{22}^{\infty})\sin\gamma\cos\gamma$ are the farfield stresses resolved on the crack line. For the isotropic case, $Q = 1/(4\pi C_{11})$. Note that the first term in Eq. (4.2.1) is associated with the applied farfield loads, and the second term represents the effects of the edge dislocation.

The potential functions due to $b_x^{\mathrm{c}}(x)$ are as follows:

$$\Phi_M(z_1; z^0) = \Phi^{\infty}(z_1) + \Phi^M(z_1; z^0) \tag{4.2.4a}$$

and

$$\Psi_M(z_2; z^0) = \Psi^{\infty}(z_2) + \Psi^M(z_2; z^0), \tag{4.2.4b}$$

where

$$\Phi^{\infty}(z_1) = \frac{-\sigma_{xy}^{\mathrm{eq.}}}{2\pi i C_{11} Q}\frac{s_1}{\tilde{s}_1}\left(1 - \frac{z_1}{\sqrt{z_1^2 - a^2}}\right), \tag{4.2.4c}$$

$$\Psi^{\infty}(z_2) = \frac{-\sigma_{xy}^{\mathrm{eq.}}}{2\pi i C_{11} Q}\frac{s_2}{\tilde{s}_2}\left(1 - \frac{z_2}{\sqrt{z_2^2 - a^2}}\right), \tag{4.2.4d}$$

$$\Phi^M(z_1; z^0) = \frac{1}{(2\pi C_{11})^2 Q}\frac{s_1}{\tilde{s}_1}[V_{11} + V_{12} - V_{13} - V_{14}] \tag{4.2.4e}$$

and

$$\Psi^M(z_2; z^0) = \frac{1}{(2\pi C_{11})^2 Q}\frac{s_2}{\tilde{s}_2}[V_{21} + V_{22} - V_{23} - V_{24}]. \tag{4.2.4f}$$

The functions V_{ij} are

$$V_{ij} = (s_j + \mu)\frac{s_j b_x^0 - b_y^0}{\tilde{s}_j}\left[\frac{1}{\sqrt{z_i^2 - a^2}} + \frac{1}{z_i - z_j^0}\left\{\frac{\sqrt{z_j^{0\,2} - a^2}}{\sqrt{z_i^2 - a^2}} - 1\right\}\right] \tag{4.2.5a}$$

and

$$V_{ij} = (s_j + \mu)\frac{s_j b_x^0 - b_y^0}{\tilde{s}_j}\left[\frac{1}{z_i - z_j^0}\left\{\frac{\sqrt{z_j^{0^2} - a^2}}{\sqrt{z_i^2 - a^2}} - 1\right\} + \frac{1}{z_i + z_j^0}\left\{\frac{\sqrt{z_j^{0^2} - a^2}}{\sqrt{z_i^2 - a^2}} + 1\right\}\right] \tag{4.2.5b}$$

for CASE #1 and CASE #2, respectively. Note that the potential functions in Eqs. (4.2.4c) and (4.2.4d) are associated with the applied farfield loads, and those in Eqs. (4.2.4e) and (4.2.4f) correspond to the edge dislocations. These Green's functions facilitate solving problems such as a closed frictional crack with kinks (wings) emanating from its tips.

Let us, now, find the stress intensity factors at the crack tip for the case when no single or double edge dislocations are present; see Eqs. (4.2.4a) and (4.2.4b). Substitution of Eqs. (4.2.4c) and (4.2.4d) in Eqs. (2.5b) and (2.5c) gives the stresses and, consequently (from Appendix D1), the SIFs,

$$K_{\mathrm{I}} = -\sqrt{\pi a}\left(\sigma_{xy}^{\infty} + \tau_{\mathrm{c}} - \mu\sigma_{yy}^{\infty}\right)\left(\frac{\Delta_2}{\Delta_3 + \mu\Delta_2}\right) \tag{4.2.6a}$$

and

$$K_{\mathrm{II}} = \sqrt{\pi a}\left(\sigma_{xy}^{\infty} + \tau_{\mathrm{c}} - \mu\sigma_{yy}^{\infty}\right)\left(\frac{\Delta_3}{\Delta_3 + \mu\Delta_2}\right). \tag{4.2.6b}$$

As is evident from Eqs. (4.2.6a) and (4.2.6b), SIFs are, in general, functions of the material properties; see Appendix A. However, when the material is orthotropic and the body- and material-coordinate systems coincide (crack lies on one of the material axis), then $\Delta_2 = 0$, and Eqs. (4.2.6a) and (4.2.6b) reduces to

$$K_{\mathrm{I}} = 0 \quad \text{and} \quad K_{\mathrm{II}} = \sqrt{\pi a}\left(\sigma_{xy}^{\infty} + \tau_{\mathrm{c}} - \mu\sigma_{yy}^{\infty}\right). \tag{4.2.7}$$

Therefore, for this special case, the SIFs are not dependent upon the material properties and they are identical to the SIFs of the isotropic case; for comparison, see SIFs of Section 4.1. Nevertheless, the SIFs are strongly affected by the conditions of the crack surfaces which are characterized by μ and τ_{c}. It is interesting that, according to Eq. (4.2.7), a crack on one material axis does not extend collinearly, but kinks; it can be shown that the kinking occurs at an angle of about 70 degrees with respect to the pre-existing crack. Finally, note that the formulation given in this section is simpler, better structured and more systematic than the ones for isotropic media given by, e.g., Horii and Nemat-Nasser (1985).

4.3. Dislocated crack

Consider plane deformations of a cracked anisotropic body. Under compressive farfield loads, a Mode-II crack may initiate Mode-I cracks (wing cracks) from the tips of a pre-existing crack. For the case of isotropic material, Nemat-Nasser and Obata (1988) used a dislocated crack to model a wing crack initiating from the tips of an existing sliding crack. In Section 4.3.1, for an open crack, we extend their formulation to the anisotropic case, and then the case of a partially-closed dislocated crack is formulated in Section 4.3.2.

4.3.1. A fully open crack dislocated at its right tip

Assume that the right tip of the crack is dislocated by d_x and d_y, as shown in Fig. 5. The boundary conditions, for an open dislocated crack, are

$$\sigma_{yy} = 0, \quad \sigma_{xy} = 0, \quad \text{for } -a \le x \le +a \text{ (free crack surfaces)}, \tag{4.3.1a}$$

$$U_x(a) = d_x, \quad U_y(a) = d_y, \tag{4.3.1b}$$

$$U_x = 0, \quad U_y = 0 \quad \text{for } x \notin [-a, +a] \tag{4.3.1c}$$

In order to find the stress functions and the crack opening displacement (COD) for this problem, the method of distributed edge dislocations is applied. The conditions for the stress-free crack surfaces lead to the following integral equations:

$$\int_{-a}^{a} \frac{1}{x-t}\{\Delta_2 b_x(t) - \Delta_1 b_y(t)\}\mathrm{d}t = 0, \quad \int_{-a}^{a} \frac{1}{x-t}\{-\Delta_3 b_x(t) - \Delta_2 b_y(t)\}\mathrm{d}t = 0. \tag{4.3.2}$$

With the aid of Eqs. (B1a), (B1b) and formulae given in Appendix A, the solution of this system of integral equations is

$$b_x(x) = -\frac{d_x}{\pi}\frac{1}{\sqrt{a^2 - x^2}}, \quad b_y(x) = -\frac{d_y}{\pi}\frac{1}{\sqrt{a^2 - x^2}}. \tag{4.3.3}$$

Consequently, from Eqs. (C4a,b), (2.10a), (2.10b), the COD and the stress functions are obtained as

$$U_x(x) = \frac{d_x}{\pi}\left[\sin^{-1}\frac{x}{a} + \frac{\pi}{2}\right], \quad U_y(x) = \frac{d_y}{\pi}\left[\sin^{-1}\frac{x}{a} + \frac{\pi}{2}\right], \tag{4.3.4a}$$

$$\Phi(z_1) = \frac{-1}{2\pi i C_{11}}\frac{1}{\tilde{s}_1}\frac{s_1 d_x - d_y}{\sqrt{z_1^2 - a^2}}, \quad \Psi(z_2) = \frac{-1}{2\pi i C_{11}}\frac{1}{\tilde{s}_2}\frac{s_2 d_x - d_y}{\sqrt{z_2^2 - a^2}}. \tag{4.3.4b}$$

Note that the dislocation density functions and the COD are not function of the material properties. Finally, the stress intensity factors at the right crack tip are given by

$$K_{\mathrm{I}} = \frac{-1}{C_{11}\sqrt{\pi a}}\left\{\frac{\Delta_2 d_x + \Delta_1 d_y}{\Delta_0}\right\}, \quad K_{\mathrm{II}} = \frac{1}{C_{11}\sqrt{\pi a}}\left\{\frac{\Delta_3 d_x - \Delta_2 d_y}{\Delta_0}\right\}. \tag{4.3.5}$$

4.3.2. A partially closed crack dislocated at its right tip under farfield loads

Consider the problem of Section 4.3.1. In addition to the right-tip dislocations d_x and d_y, the body is subjected to the uniform farfield loads σ_{yy}^{∞} and σ_{xy}^{∞}. Assume that the combination of the farfield loads and the right-tip dislocation renders the crack partly closed on its left side, from $x=-a$ to some point

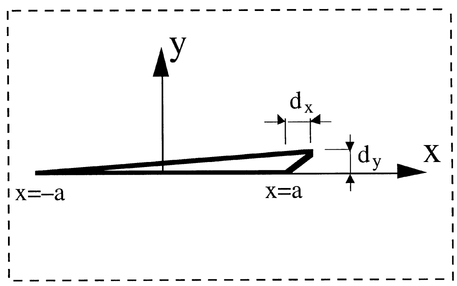

Fig. 5. An open crack dislocated at its right tip by $d = d_x + id_y$.

$x=b$ which must be computed; see Fig. 6. These conditions require,

$$\sigma_{yy} = \sigma^{c}_{yy} = 0, \quad \sigma_{xy} = \sigma^{c}_{xy} = 0, \quad \text{for } b \le x \le +a, \tag{4.3.6a}$$

$$U_x(a) = d_x, \quad U_y(a) = d_y, \tag{4.3.6b}$$

$$\sigma_{yy} = \sigma^{c}_{yy} + \sigma^{0}_{yy} + \sigma^{\infty}_{yy} = 0, \quad \sigma_{xy} = \sigma^{c}_{xy} + \sigma^{\infty}_{yy} = 0, \quad U_y = 0 \quad \text{for } -a \le x \le b, \tag{4.3.6c}$$

$$U_x = 0 \quad \text{and} \quad U_y = 0 \quad \text{for } x \notin [-a, +a], \tag{4.3.6d}$$

where σ^{c}_{yy} and σ^{c}_{xy} are the stresses due to the presence of the distributed dislocations along the crack, and σ^{0}_{yy} is the normal stress transmitted across the crack surfaces over the contact region. The stress σ^{0}_{yy} and the length of the contact surfaces (the location of point b) are unknown and must be determined as part of the solution.

The stress boundary conditions (Eqs. (4.3.6c)) lead to the following integral equations:

$$\frac{1}{\pi C_{11}\Delta_0}\int_{-a}^{a}\frac{1}{x-t}\{\Delta_2 b_x(t) - \Delta_1 b_y(t)H(t-b)\}\mathrm{d}t + \sigma^{\infty}_{yy} + \sigma^{0}_{yy}H(b-x) = 0 \tag{4.3.7a}$$

and

$$\frac{1}{\pi C_{11}\Delta_0}\int_{-a}^{a}\frac{1}{x-t}\{-\Delta_3 b_x(t) + \Delta_2 b_y(t)H(t-b)\}\mathrm{d}t + \sigma^{\infty}_{xy} = 0, \tag{4.3.7b}$$

where $H(s)$ is unity or zero for positive or negative s, respectively. In order to obtain the length of the contact zone (point b) and the contact stress $\sigma^{0}_{yy}(x)$, we note the conditions given in Eqs. (4.3.6b,c) and (4.3.6d). The normal contact stress $\sigma^{0}_{yy}(x)$ is bounded at $x=b$ and $x=-a$, and $b_x(x)$ is bounded at $x=b$ but is singular at $x=-a$. The final results are

$$b = a - \frac{2}{\pi C_{11}}\frac{d_y}{\Delta_3\sigma^{\infty}_{yy} + \Delta_2\sigma^{\infty}_{xy}}, \quad \sigma^{0}_{yy}(x) = \frac{\Delta_3\sigma^{\infty}_{yy} + \Delta_2\sigma^{\infty}_{xy}}{\Delta_3}\frac{\sqrt{b-x}}{\sqrt{a-x}}. \tag{4.3.8}$$

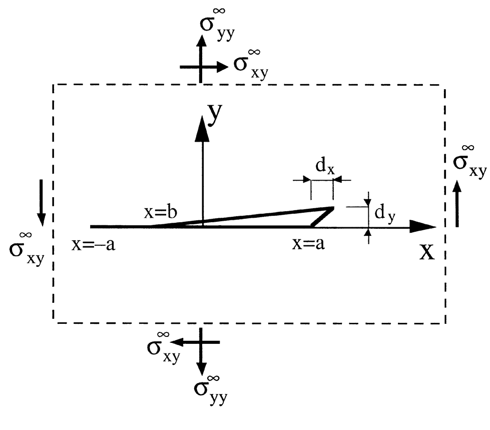

Fig. 6. A crack is first dislocated at its right tip by $d=d_x+id_y$, and then subjected to farfield loads which cause partial crack closure (from $x=-a$ to $x=b$).

For the contact point to lie on the crack line and the normal contact stress be negative (compressive), the following condition must be satisfied:

$$\left(\Delta_3\sigma_{yy}^{\infty} + \Delta_2\sigma_{xy}^{\infty}\right) > 0 \Leftrightarrow \left(\beta_1\left(\alpha_2^2 + \beta_2^2\right) + \beta_2\left(\alpha_1^2 + \beta_1^2\right)\right)\sigma_{yy}^{\infty} + \left(\alpha_1\beta_2 + \alpha_2\beta_1\right)\sigma_{xy}^{\infty} < 0. \tag{4.3.9}$$

Compare this condition with the one given in Eq. (4.1.15).

Now, use Eq. (4.3.8) in Eqs. (4.3.7a) and (4.3.7b) and then solve the integral equations (Eqs. (4.3.7a) and (4.3.7b)) for the dislocation densities, $b_x(x)$ and $b_y(x)$, to arrive at

$$b_x(x) = C_{11}\frac{\Delta_0}{\Delta_3} - \frac{1}{\sqrt{a^2 - x^2}}\left[\sigma_{xy}^{\infty}x + \frac{1}{\pi C_{11}\Delta_0}(\Delta_3 d_x - \Delta_2 d_y)\right] - C_{11}\frac{\Delta_2}{\Delta_3}\big(\Delta_3\sigma_{yy}^{\infty} + \Delta_2\sigma_{xy}^{\infty}\big)\frac{\sqrt{x-b}}{\sqrt{a-x}}H(x-b) \tag{4.3.10a}$$

and

$$b_y(x) = -C_{11}\big(\Delta_3\sigma_{yy}^{\infty} + \Delta_2\sigma_{xy}^{\infty}\big)\frac{\sqrt{x-b}}{\sqrt{a-x}}H(x-b). \tag{4.3.10b}$$

The COD components are

$$\begin{aligned} U_x(x) = {} & -C_{11}\frac{\Delta_4}{\Delta_3}\sigma_{xy}^{\infty}\sqrt{a^2 - x^2} + \frac{1}{\pi}\left(d_x - \frac{\Delta_2}{\Delta_3}d_y\right)\left\{\sin^{-1}\frac{x}{a} + \frac{\pi}{2}\right\} \\ & - C_{11}\frac{\Delta_2}{\Delta_3}\big(\Delta_3\sigma_{yy}^{\infty} + \Delta_2\sigma_{xy}^{\infty}\big)\sqrt{(a-x)(x-b)}H(x-b) \\ & + \frac{\Delta_2}{\Delta_3}\frac{d_y}{\pi}\left\{\sin^{-1}\left[\frac{x-b-(a-x)}{a-b}\right] + \frac{\pi}{2}\right\}H(x-b) \end{aligned} \tag{4.3.11a}$$

$$\begin{aligned} U_y(x) = {} & -C_{11}\big(\Delta_3\sigma_{yy}^{\infty} + \Delta_2\sigma_{xy}^{\infty}\big)\sqrt{(a-x)(x-b)}H(x-b) \\ & + \frac{d_y}{\pi}\left\{\sin^{-1}\left[\frac{x-b-(a-x)}{a-b}\right] + \frac{\pi}{2}\right\}H(x-b). \end{aligned} \tag{4.3.11b}$$

4.4. A crack and self-equilibrating tractions

Consider plane deformations of an anisotropic body containing a crack. A segment of the upper and lower crack faces ($b \le x \le c$) are loaded by self-equilibrating (i.e., equal in magnitude but opposite in sign) normal and shear tractions such that the crack faces are not in contact; see Fig. 7. We investigate this problem for the cases of:

1. loading on a part of the crack faces,
2. loading on the entire crack faces, and
3. loading just at an arbitrary point on the crack faces by concentrated forces.

On the crack faces, the boundary conditions are

$$\sigma^{c}_{yy} + \sigma^{0}_{yy} = 0, \quad \sigma^{c}_{xy} + \sigma^{0}_{xy} = 0. \tag{4.4.1}$$

Terms with superscript 'c' represent the stresses corresponding to the distributed edge dislocations, $b(x) = b_x(x) + ib_y(x)$, along the crack line, and σ^0_{yy} and σ^0_{xy} are the self-equilibrating tractions externally applied on the crack faces. As in Section 4.1, these BCs result in a set of singular integral equations for $b_x(x)$ and $b_y(x)$. Using the Δ_j's of Appendix A and of Eqs. (B1a) and (B1b), the solution of the system of integral equations is

$$\frac{1}{\pi\Delta_0 C_{11}}\{\Delta_2 b_x(x) - \Delta_1 b_y(x)\} = \frac{1}{\pi^2\sqrt{a^2-x^2}}\int_{-a}^{+a}\frac{\sqrt{a^2-t^2}}{x-t}\sigma^0_{yy}(t)\mathrm{d}t \tag{4.4.2a}$$

and

$$\frac{1}{\pi\Delta_0 C_{11}}\{-\Delta_3 b_x(x) + \Delta_2 b_y(x)\} = \frac{1}{\pi^2\sqrt{a^2-x^2}}\int_{-a}^{+a}\frac{\sqrt{a^2-t^2}}{x-t}\sigma^0_{xy}(t)\mathrm{d}t. \tag{4.4.2b}$$

For the case of constant tractions, i.e.,

$$\sigma^0_{yy} = p \quad \text{and} \quad \sigma^0_{xy} = q \quad \text{for } b \le x \le c \text{ (and elsewhere, zero)}, \tag{4.4.3}$$

the integral in Eq. (4.4.2a) becomes

$$\int_{-a}^{+a}\frac{\sqrt{a^2-t^2}}{x-t}\sigma^0_{yy}\,\mathrm{d}t = p\int_{b}^{c}\frac{\sqrt{a^2-t^2}}{x-t}\,\mathrm{d}t = pI(a,\,b,\,c,\,x), \tag{4.4.4}$$

with

$$I = \sqrt{a^2-b^2} - \sqrt{a^2-c^2} + x\left[\sin^{-1}\frac{c}{a} - \sin^{-1}\frac{b}{a}\right] + \log\left[\frac{(a^2+\sqrt{a^2-c^2}\sqrt{a^2-x^2}-cx)(x-b)}{(a^2+\sqrt{a^2-b^2}\sqrt{a^2-x^2}-bx)(x-c)}\right].$$

Now, the dislocation density functions, from Eqs. (4.4.2a) and (4.4.2b), are

$$b_x(x) = \frac{C_{11}}{\pi}\{-\Delta_2 p + \Delta_1 q\}\frac{I(a,\,b,\,c,\,x)}{\sqrt{a^2-x^2}}, \quad b_y(x) = \frac{C_{11}}{\pi}\{-\Delta_3 p - \Delta_2 q\}\frac{I(a,\,b,\,c,\,x)}{\sqrt{a^2-x^2}}. \tag{4.4.5}$$

Next, we calculate the associated potential functions using Eqs. (2.10a), (2.10b), (4.4.4), (4.4.5),

$$\Phi(z_1) = \frac{-1}{2\pi^2}\frac{s_2 p + q}{s_1 - s_2}\int_{-a}^{a}\frac{I(a,\,b,\,c,\,t)}{(z_1-t)\sqrt{a^2-t^2}}\,\mathrm{d}t \tag{4.4.6a}$$

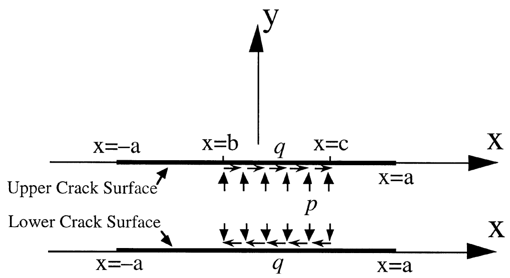

Fig. 7. A balanced loading of the crack faces; normal and tangential tractions, p and q, act on the segment $b \le x \le c$.

and

$$\Psi(z_2) = \frac{1}{2\pi^2}\frac{s_1 p + q}{s_1 - s_2}\int_{-a}^{a}\frac{I(a, b, c, t)}{(z_2 - t)\sqrt{a^2 - t^2}}\,\mathrm{d}t, \tag{4.4.6b}$$

where the definite integral is expressed as

$$\int_{-a}^{a}\frac{I(a, b, c, t)}{(z - t)\sqrt{a^2 - t^2}}\,\mathrm{d}t = \pi\frac{\sqrt{a^2 - b^2} - \sqrt{a^2 - c^2}}{\sqrt{z^2 - a^2}} - \pi\left(\sin^{-1}\frac{c}{a} - \sin^{-1}\frac{b}{a}\right)\left(1 - \frac{z}{\sqrt{z^2 - a^2}}\right) + \int_{-a}^{a}\frac{\log\left[\dfrac{(a^2 + \sqrt{a^2 - c^2}\sqrt{a^2 - t^2} - ct)|t - b|}{(a^2 + \sqrt{a^2 - b^2}\sqrt{a^2 - t^2} - bt)|t - c|}\right]}{z - t}\,\mathrm{d}t. \tag{4.4.7}$$

The last part of this integral seems to be difficult to be carried out. Thus, for this particular loading case, we evaluate the potential functions by the method of Section 3, the Hilbert problem. This is given at the end of this section.

Consider the solution when the entire crack faces are loaded uniformly by p and q. For this, set $b = -a$ and $c = +a$ to obtain $I(a, b, c, x) = \pi x$. Thus, the dislocation functions and the potentials, using Eq. (B2c), become

$$b_x(x) = C_{11}\{-\Delta_2 p + \Delta_1 q\}\frac{x}{\sqrt{a^2 - x^2}}, \quad b_y(x) = C_{11}\{-\Delta_3 p - \Delta_2 q\}\frac{x}{\sqrt{a^2 - x^2}}, \tag{4.4.8}$$

$$\Phi(z_1) = \frac{ps_2 + q}{2(s_1 - s_2)}\left\{1 - \frac{z_1}{\sqrt{z_1^2 - a^2}}\right\}, \quad \Psi(z_2) = \frac{ps_1 + q}{2(s_2 - s_1)}\left\{1 - \frac{z_2}{\sqrt{z_2^2 - a^2}}\right\}. \tag{4.4.8b}$$

Note that these functions are similar to Eqs. (4.1.7a), (4.1.7b), (4.1.10) of Section 4.1.

Finally, consider the case of a point force (concentrated normal and shear forces) applied at the arbitrary point $x = \zeta$ on the crack face. Considering $b = \zeta - \varepsilon$ and $c = \zeta + \varepsilon$, with $\varepsilon \ll 1$, one can represent the concentrated forces as

$$F_x = [2\varepsilon q]_{\varepsilon\to 0}, \quad F_y = [2\varepsilon p]_{\varepsilon\to 0} \quad (F_x \text{ and } F_y \text{ are applied at } x = \zeta). \tag{4.4.9}$$

This now leads to

$$I(a, b, c, x) = I(a, \zeta, \varepsilon, x) = 2\varepsilon\frac{\sqrt{a^2 - \zeta^2}}{x - \zeta} + O(\varepsilon^2). \tag{4.4.10}$$

Now, use Eq. (4.4.10) in Eqs. (4.4.5), (4.4.6a) and (4.4.6b), as well as Eq. (B2f), to arrive at

$$b_x(x) = \frac{C_{11}}{\pi}\{-\Delta_2 F_y + \Delta_1 F_x\}\frac{\sqrt{a^2 - \zeta^2}}{(x - \zeta)\sqrt{a^2 - x^2}}, \tag{4.4.11a}$$

$$b_y(x) = \frac{C_{11}}{\pi}\{-\Delta_3 F_y - \Delta_2 F_x\}\frac{\sqrt{a^2 - \zeta^2}}{(x - \zeta)\sqrt{a^2 - x^2}}, \tag{4.4.11b}$$

$$\Phi(z_1) = \frac{-1}{2\pi}\frac{s_2F_y + F_x}{s_1 - s_2}\left\{\frac{\sqrt{a^2-\zeta^2}}{(z_1-\zeta)\sqrt{z_1^2-a^2}}\right\}, \quad (\text{for } z_1 \notin [-a, a]) \tag{4.4.12a}$$

and

$$\Psi(z_2) = \frac{1}{2\pi}\frac{s_1F_y + F_x}{s_1 - s_2}\left\{\frac{\sqrt{a^2-\zeta^2}}{(z_2-\zeta)\sqrt{z_2^2-a^2}}\right\} \quad (\text{for } z_2 \notin [-a, a]). \tag{4.4.12b}$$

Note that for this case, there is an additional singularity at point $x=\zeta$ where the point force is applied; this singularity is stronger than the square-root singularities occurring at the crack tips, $x=-a$ and $x=a$.

Having calculated the potential functions, the stresses are readily computed using Eqs. (2.5b) and (2.5c). Then, the SIFs, K_{I} and K_{II}, at the right tip of the crack are

$$K_{\mathrm{I}} = \frac{F_y}{\sqrt{\pi a}}\sqrt{\frac{a+\zeta}{a-\zeta}} \quad \text{and} \quad K_{\mathrm{II}} = \frac{F_x}{\sqrt{\pi a}}\sqrt{\frac{a+\zeta}{a-\zeta}}. \tag{4.4.13}$$

These are also the Green functions for calculating the SIFs for an open crack subjected to self-equilibrating tractions, $\sigma^0_{yy}(x)$ and $\sigma^0_{xy}(x)$ $(b \le x \le c)$,

$$K_{\mathrm{I}} = \frac{1}{\sqrt{\pi a}}\int_b^c \sqrt{\frac{a+\zeta}{a-\zeta}}\sigma^0_{yy}(\zeta)\mathrm{d}\zeta \quad \text{and} \quad K_{\mathrm{II}} = \frac{1}{\sqrt{\pi a}}\int_b^c \sqrt{\frac{a+\zeta}{a-\zeta}}\sigma^0_{xy}(\zeta)\mathrm{d}\zeta. \tag{4.4.14}$$

Note that the SIFs given in Eq. (4.4.13) are not functions of the material properties, thus, they are *identical* to the ones for the *isotropic* case. This is because the applied concentrated forces are *balanced* on the crack faces (Barnett and Asaro, 1972). See Section 3.3 for the case when the concentrated forces are applied on just one crack face (the corresponding SIFs then *do* depend on the material properties).

Referring to Eqs. (4.4.6a), (4.4.6b) and (4.4.7), consider the calculation of the stress functions using the Hilbert method of Section 3.1. For the self-equilibrating tractions on the crack faces, Eqs. (3.1.3a) and (3.1.3b) reduce to

$$(1+\alpha s_1)\Theta(z) + (1+\alpha s_3)\Omega(z) = -\frac{p-\alpha q}{\pi i}\frac{1}{\sqrt{z^2-a^2}}\left\{\int_b^c \frac{\sqrt{t^2-a^2}}{t-z}\,\mathrm{d}t + R(z)\right\} \tag{4.4.15a}$$

and

$$(1+\alpha s_1)\Theta(z) - (1+\alpha s_3)\Omega(z) = 0. \tag{4.4.15b}$$

Here, the single-valuedness of the displacement field requires that $R(z)=C_0=0$. Using this and a procedure similar to Section 3, we obtain

$$\Theta(z) = -\frac{p-\alpha q}{2\pi}\frac{1}{1+\alpha s_1}H(a, b, c, z), \qquad \Omega(z) = -\frac{p-\alpha q}{2\pi}\frac{1}{1+\alpha s_3}H(a, b, c, z), \tag{4.4.16a}$$

$$\Phi(z) = \frac{1}{2\pi}\frac{ps_2+q}{s_1-s_2}H(a, b, c, z), \qquad \Psi(z) = \frac{1}{2\pi}\frac{ps_1+q}{s_2-s_1}H(a, b, c, z), \tag{4.4.16b}$$

$$K_{\rm I} = \frac{-p}{\sqrt{\pi a}} M_-(a, b, c), \qquad K_{\rm II} = \frac{-q}{\sqrt{\pi a}} M_-(a, b, c) \quad \text{at } x = a, \tag{4.4.17a}$$

$$K_{\rm I} = \frac{p}{\sqrt{\pi a}} M_+(a, b, c) \qquad \text{and} \qquad K_{\rm II} = \frac{q}{\sqrt{\pi a}} M_+(a, b, c) \quad \text{at } x = -a, \tag{4.4.17b}$$

where the functions H and M are defined in Section 3.1; see Eqs. (3.1.5c) and (3.1.14c).

Comparing Eq. (4.4.16b) with Eqs. (4.4.6a) and (4.4.6b) along with Eq. (4.4.7), we obtain an expression for the definite integral in Eq. (4.4.7),

$$\int_{-a}^{a} \frac{\log\left[\dfrac{(a^2 + \sqrt{a^2 - c^2}\sqrt{a^2 - t^2} - ct)|t - b|}{(a^2 + \sqrt{a^2 - b^2}\sqrt{a^2 - t^2} - bt)|t - c|}\right]}{z - t}\,{\rm d}t$$

$$= -\pi\left[\tan^{-1}\frac{cz - a^2}{\sqrt{a^2 - c^2}\sqrt{z^2 - a^2}} - \tan^{-1}\frac{bz - a^2}{\sqrt{a^2 - b^2}\sqrt{z^2 - a^2}} - \sin^{-1}\frac{c}{a} + \sin^{-1}\frac{b}{a}\right]. \tag{4.4.18}$$

4.5. Non-aligned periodic open cracks under farfield loads

Consider an infinite set of straight cracks with a common length $2a$ in an infinitely extended anisotropic plate. The cracks are parallel and spaced by h and d in the x-direction and y-direction, respectively, as shown in Fig. 8. The farfield load (σ_{yy}^{∞} and σ_{xy}^{∞}) is such that the cracks remain open. Some exact and approximation formulae are available for the isotropic case and special geometries, e.g., collinear cracks with $d = 0$ and $h > 2a$, and parallel cracks with $h = 0$ and $d \neq 0$; see Murakami (1987). Here, we consider a more general case for which the material is anisotropic, and d and h are, in general, non-zero, summarizing the key results in what follows.

The stress-free crack-face conditions for a typical crack (zero-th crack of Fig. 8) are

$$\sigma_{yy}^{\rm c}(x, 0) + \sigma_{yy}^{\infty} = 0 \quad \text{and} \quad \sigma_{xy}^{\rm c}(x, 0) + \sigma_{xy}^{\infty} = 0, \tag{4.5.1}$$

where the first terms in Eq. (4.5.1) are the stresses on the line of the zero-th crack corresponding to all the cracks. Each crack is modeled by edge dislocations b_x and b_y distributed along its line; because of periodicity the dislocation functions b_x and b_y are the same for all cracks. The final form of Eq. (4.5.1) is

$$\begin{aligned} &2\,{\rm Real}\left[A_1 \int_{-a}^{a} [s_1 b_x(t) - b_y(t)]{\rm cotan}[B_1(x - t)]{\rm d}t\right] \\ &\quad + 2\,{\rm Real}\left[A_2 \int_{-a}^{a} [s_2 b_x(t) - b_y(t)]{\rm cotan}[B_2(x - t)]{\rm d}t\right] + \sigma_{yy}^{\infty} = 0 \end{aligned} \tag{4.5.2a}$$

and

$$\begin{aligned} &-2\,{\rm Real}\left[A_1 s_1 \int_{-a}^{a} [s_1 b_x(t) - b_y(t)]{\rm cotan}[B_1(x - t)]{\rm d}t\right] \\ &\quad - 2\,{\rm Real}\left[A_2 s_2 \int_{-a}^{a} [s_2 b_x(t) - b_y(t)]{\rm cotan}[B_2(x - t)]{\rm d}t\right] + \sigma_{xy}^{\infty} = 0, \end{aligned} \tag{4.5.2b}$$

where

$$A_j = \frac{1}{2\pi i C_{11}} \frac{-\pi}{\tilde{s}_j P_j}, \quad B_j = \frac{-\pi}{P_j}, \quad P_j = h + s_j d, \quad \text{for} \quad j = 1, 2. \tag{4.5.2c}$$

These are two integral equations for the two unknowns, $b_x(x)$ and $b_y(x)$, $-a \le x \le a$. A closed form solution for $b_x(x)$ and $b_y(x)$ is difficult to obtain. When the material is isotropic and the cracks are collinear, the following formula (Westergaard, 1939) gives the SIF:

$$K_{\rm I} = \sqrt{\frac{h}{\pi a} \tan\left(\frac{\pi a}{h}\right)} \sigma_{yy}^{\infty} \sqrt{\pi a} \quad \text{for} \quad h > 2a \quad \text{and} \quad d = 0. \tag{4.5.3}$$

Again, for the isotropic case and when the cracks are parallel (with $h = 0$ and $d \neq 0$), Yokobori and Ichikawa (1967) propose an approximation formula for SIF. The more general case shown in Fig. 8 seems not to have been considered, even for the isotropic case. Next, consider solving this problem.

The system of coupled singular integral equations in Eqs. (4.5.2a,b) can be solved numerically, as outlined in Appendix C. The auxiliary conditions for this problem are of the type in Eqs. (C3a) and (C3b); these two conditions ensure that each crack is closed at both ends. The numerical solution gives $B_x(x)$ and $B_y(x)$ along the crack line and, in particular, $B_x(x=a)$ and $B_y(x=a)$. Only the last two parameters are required for the calculation of the SIFs at the crack tip; see Appendix D. The SIFs are

$$K_{\rm I} = \sqrt{\frac{\pi}{a}} \left\{ \frac{\Delta_2 B_x(a) + \Delta_1 B_y(a)}{\Delta_0 C_{11}} \right\}, \quad K_{\rm II} = \sqrt{\frac{\pi}{a}} \left\{ \frac{-\Delta_3 B_x(a) + \Delta_2 B_y(a)}{\Delta_0 C_{11}} \right\}, \tag{4.5.4}$$

where the Δ_j's are defined in Appendix A. For the isotropic case, these formulae reduce to $K_{\rm I} = B_y(a)\sqrt{\pi/a}/(4C_{11})$ and $K_{\rm II} = B_x(a)\sqrt{\pi/a}/(4C_{11})$. The numerical procedure outlined in Appendix C gives the values of $B_x(x)$ and $B_y(x)$ to any desired degree of accuracy, i.e. by a finer partitioning the interval $[-a, +a]$. The numerical results may be used to estimate the error involved in the following

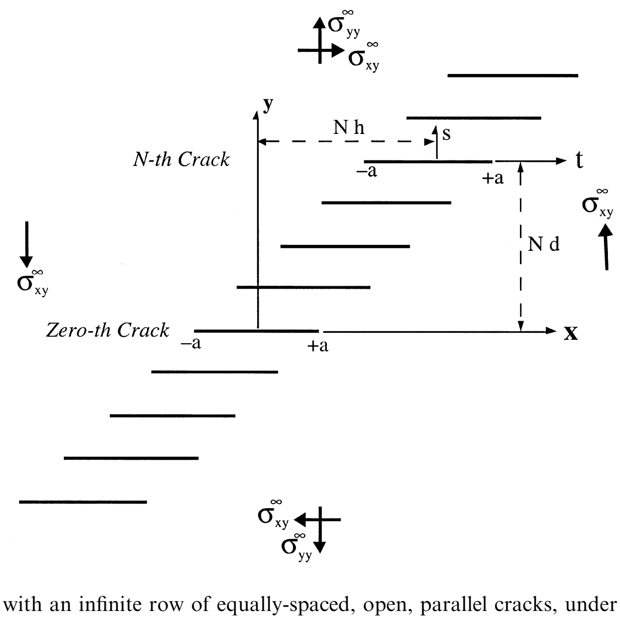

Fig. 8. A body with an infinite row of equally-spaced, open, parallel cracks, under farfield stresses.

approximation calculation of $B_x(x=a)$ and $B_y(x=a)$:

$$B_x(a) \approx \frac{-\sigma_{yy}^{\infty} A_{22} + \sigma_{xy}^{\infty} A_{12}}{A_{11}A_{22} - A_{12}A_{21}} \quad \text{and} \quad B_y(a) \approx \frac{-\sigma_{xy}^{\infty} A_{11} + \sigma_{yy}^{\infty} A_{21}}{A_{11}A_{22} - A_{12}A_{21}}, \tag{4.5.5a}$$

with

$$A_{11} = 2\,\mathrm{Real}\left[\frac{\pi}{2\pi i C_{11}}\left\{\frac{s_1}{\tilde{s}_1}Q_1 + \frac{s_2}{\tilde{s}_2}Q_2\right\}\right], \quad A_{12} = 2\,\mathrm{Real}\left[\frac{-\pi}{2\pi i C_{11}}\left\{\frac{1}{\tilde{s}_1}Q_1 + \frac{1}{\tilde{s}_2}Q_2\right\}\right], \tag{4.5.5b}$$

$$A_{21} = 2\,\mathrm{Real}\left[\frac{-\pi}{2\pi i C_{11}}\left\{\frac{s_1^2}{\tilde{s}_1}Q_1 + \frac{s_2^2}{\tilde{s}_2}Q_2\right\}\right], \quad A_{22} = A_{11} \tag{4.5.5c}$$

and

$$Q_j = \frac{B_j}{\pi}\int_{-a}^{a} \mathrm{cotan}[B_j(a-t)]\sqrt{a^2 - t^2}\,\mathrm{d}t = \sum_{N=-\infty}^{N=+\infty}\left(W_j - \sqrt{W_j^2 - a^2}\right), \quad W_j = a - Np_j. \tag{4.5.5d}$$

In order to examine the range of validity of the above approximate formula, the SIFs are calculated for different elastic constants (E_{11}, E_{22}, G_{12}, ν_{12} and β), the horizontal and vertical distances between the crack centers (h and d), and the farfield loads (σ_{yy}^{∞} and σ_{xy}^{∞}). Specific values are assigned to the elastic moduli and the farfield loads, as shown in Table 1. Due to linearity, the results can be adjusted for other material constants and loads. The calculated SIFs are presented in Table 1. It is rather difficult to establish a parameter for estimation of the error involved in the approximation formula. This is because all the above parameters (mentioned in the previous paragraph) may contribute to this error. However, from Eqs. (4.5.5a–d), it seems that the parameter, $p_j/2a = h/2a + s_j\, d/2a$, might be a good measure of the error; this parameter combines the geometry and the elastic constants. Since this parameter is a complex number, for simplicity, we establish the error range based only on the geometry, i.e., $h/2a$ and $d/2a$.

The cases considered in Table 1 show that, for $d/2a > 2$ and $h/2a > 1.5$, the approximate formula gives the values of the SIFs with an error of less than 2% when compared with the complete numerical solution; compare this with the cases presented in Murakami (1987) for isotropic materials where, in general, for the same range of $d/2a$ and $h/2a$, the error is higher than 2%. When the cracks are close to each other, the error can be larger than 2%, depending upon the geometry, material constants, and the load. Note that for the cases when d and h are both non-zero (even for the special case of isotropy), a far-field tensile load produces a Mode-II fracture component; see, e.g. CASE #8 of Table 1, which shows a non-zero K_{II}.

As an extension to this section, consider the practical problem of non-aligned periodic closed cracks; the solution to this problem is obtained using the results presented in Section 4.2 and this section.

5. Method of the resultant-force

Consider, again, modeling a crack by continuously distributed edge dislocations along its line. The boundary conditions can be taken into consideration either by *tractions* or by the *resultant forces* along the crack line. In either way, the resulting coupled singular integral equations may be solved numerically, e.g., collocation method (see, e.g. Erdogan et al., 1973; Gerasoulis, 1982). For illustration, consider a traction-free crack in a plate. For the collocation method, the crack line is divided into N equal sub-intervals, and the mid-points of the sub-intervals are taken as the collocation points. The

Table 1
Normalized SIFs calculated **N**umerically from Eqs. (4.5.2a,b) and **A**pproximately from Eqs. (4.5.5a–d) for various indicated elastic constants, geometrical periodicity and loads; the normalization is as follows: $\tilde{K}^{(N,\,A)}_{I,\,II} = K^{(N,\,A)}_{I,\,II}/\sqrt{\pi a(\sigma_{yy}^{\infty 2} + \sigma_{xy}^{\infty 2})}$.. For all cases, $a =$ 1 and, for isotropic cases, $E_{11}/E_{22} = 1.000001$ are used

#	E_{11}	E_{22}	G_{12}	ν_{12}	β	h	d	σ_{yy}^{∞}	σ_{xy}^{∞}	$\tilde{K}_I^{(N)}$	$\tilde{K}_I^{(A)}$	$\tilde{K}_{II}^{(N)}$	$\tilde{K}_{II}^{(A)}$
1	1	1	0.4	0.25	0	2.01	0	1	0	8.9934	7.5054	0	0
2	1	1	0.4	0.25	0	2.1	0	1		2.9865	2.7058	0	0
3	1	1	0.4	0.25	0	2.25	0	1	0	2.0154	1.9076	0	0
4	1	1	0.4	0.25	0	2.5	0	1	0	1.5651	1.5223	0	0
5	1	1	0.4	0.25	0	5	0	1	0	1.0753	1.0742	0	0
6	1	1	0.4	0.25	0	100	0	1	0	1.0002	1.0002	0	0
7	1	1	0.4	0.25	0	0	0.25	1	0	0.1994	0.0790	0	0
8	1	1	0.4	0.25	0	0.1	0.25	1	0	0.5297	0.5607	−1.0280	−1.1573
9	1	1	0.4	0.25	0	2.1	0.25	0	1	0.5025	0.4911	1.4438	1.5413
10	1	1	0.4	0.25	0	2.25	2.5	0	1	−0.1794	−0.1821	0.9342	0.9266
11	1	1	0.4	0.25	0	0	5	1	1	0.5995	0.5973	0.7491	0.7505
12	1	1	0.4	0.25	0	2.5	1	2	1	1.4011	1.4044	0.4121	0.4518
13	1	1	0.4	0.25	0	5	2	1	2	0.5033	0.5033	0.9008	0.9020
14	1	1	0.4	0.25	0	5	2	0	1	0.017	0.0175	0.9987	0.9997
15	1	1	0.4	0.25	0	100	100	1	1	0.7071	0.7071	0.7071	0.7070
16	2	1	0.4	0.25	0	2.25	0	1	0	2.0154	1.9076	0	0
17	3	1	0.4	0.35	0	0	0.5	0	1	0	0	1.7817	1.9493
18	6	1	0.6	0.30	0	2.1	0.5	1	5	0.3604	0.3771	0.8434	0.8978
19	10	1	0.6	0.30	0	2.5	0.1	5	1	1.5922	1.5434	0.5330	0.5085
20	1	2	0.8	0.20	0	100	100	1	1	0.7071	0.7071	0.7070	0.7070
21	1	5	0.8	0.40	0	100	0	10	1	0.9952	0.9952	0.0995	0.995
22	1	10	0.5	0.30	0	2.5	0	1	5	0.3069	0.2986	1.5346	1.4928
23	1	4	0.6	0.30	30	2.1	0.5	1	0	2.3045	2.2377	−0.1474	−0.392
24	1	4	0.6	0.30	30	2.1	1.0	1	0	1.7120	1.7105	−0.2167	−0.1616
25	8	1	1.0	0.25	45	0	0.2	1	4	1.3218	1.7185	3.5875	3.6153
26	1	1	0.8	0.30	60	5	10	1	1	0.6948	0.6949	0.6941	0.6940
27	1	10	0.4	0.35	90	4	2	100	1	1.1001	1.0996	−0.0327	−0.0310
28	4	1	0.4	0.35	90	100	0.1	1	1	0.7072	0.7072	0.7072	0.7072
30	2	1	0.4	0.30	−30	0	1	1	2	0.0565	−0.0298	1.3973	1.4883
31	2	1	0.4	0.30	−30	0	2	1	2	0.1839	0.1576	1.1010	1.1236
32	2	1	0.4	0.30	−30	0	3	1	2	0.2738	0.2655	1.0009	1.0076
33	2	1	0.4	0.30	−30	0	4	1	2	0.3303	0.3271	0.9580	0.9605
34	5	1	0.6	0.25	−45	2.5	5	1	0	1.0057	1.0068	−0.0629	−0.0632
35	1	2	0.8	0.25	−60	3	1	0	1	0.0412	0.0472	0.9856	0.9969
36	1	10	0.8	0.40	−90	2.1	0	1	10	0.2972	0.2692	2.9716	2.6924
37	10	1	0.6	0.30	5	0	100	1	0	0.9998	0.9998	0	0
38	1	4	0.4	0.25	−10	100	100	1	1	0.7072	0.7072	0.7071	0.7071
39	1	1	0.4	0.25	0	0	0.25	1	0	0.1994	0.0792	0	0
40	1	1	0.4	0.25	0	0	0.35	0	1	0	0	2.4533	2.6999
41	1	1	0.4	0.25	0	0	0.5	0	1	0	0	2.0944	2.3075
42	1	1	0.4	0.25	0	2.05	0.2	0	1	0.6434	0.6291	1.5652	1.7001
43	1	1	0.4	0.25	0	0	0.35	1	0	0.2361	0.1102	0	0
44	5	1	0.8	0.25	30	0	0.35	1	0	0.2762	0.1492	0.0313	0.0276
45	5	1	0.8	0.25	30	0.2	0.35	1	0	0.3660	0.2984	−1.1090	−1.3057
46	1	1	0.4	0.25	0	2.15	0.6	0	1	0.0911	0.1105	0.9961	1.0415
47	1	1	0.4	0.25	0	2.15	1.0	0	1	−0.1082	−0.0879	0.8829	0.8901
48	1	8	0.6	0.35	45	2.15	0.6	0	1	0.2232	0.2321	1.0547	1.0965

system of integral equations is then numerically solved by satisfying the traction-free conditions at the collocation points taken along the crack line. Now, consider one of these sub-intervals taken on the crack line (say, from point A to point B, i.e., the segment [A, B]). The method of traction-free boundary conditions requires that the normal and shear stresses at the collocation point within this sub-interval be zero. An alternative formulation is to require that the resultant force acting on the entire sub interval [A, B] remains zero. In this formulation, the tractions are integrated along the sub-interval analytically. While the traction method satisfies the traction condition at just one point (mid-point), the resultant-force method involves the effect of the tractions all along the entire sub-interval (Azhdari, 1995). In view of this, it is expected that the resultant-force condition would require fewer elements and less computational effort to achieve a certain desired accuracy, when compared with the traction method.

The resultant force at point (x, y) corresponding to a single edge dislocation, $b^0 = (b_x^0, b_y^0)$ at point (x^0, y^0), is obtained by combining Eqs. (2.5e), (2.10a) and (2.10b). The result is

$$f_x = 2\,\mathrm{Real}\left[\left(A_1 s_1^2 Y_1 + A_2 s_2^2 Y_2\right)b_x^0 - (A_1 s_1 Y_1 + A_2 s_2 Y_2)b_x^0\right] + c_x, \tag{5.1a}$$

$$f_y = -2\,\mathrm{Real}\left[(A_1 s_1 Y_1 + A_2 s_2 Y_2)b_x^0 - (A_1 Y_1 + A_2 Y_2)b_x^0\right] + c_y, \tag{5.1b}$$

$$Y_j = \log\left(z_j - z_j^0\right), \quad A_j = \frac{1}{2\pi i C_{11}\tilde{s}_j}, \tag{5.1c}$$

where c_x and c_y are constants to be determined.

To present the basic elements of the method, consider the case of an infinitely extended anisotropic plate containing a crack (straight or curved) subjected to some tractions on its boundaries. The resultant forces at point (x, y) corresponding to continuously distributed edge dislocations along the crack line, are

$$F_x(x, y) = \int_{-a}^{a} f_x(x, y; x^0, y^0)\,\mathrm{d}s(x^0, y^0) + C_x \tag{5.2a}$$

and

$$F_y(x, y) = \int_{-a}^{a} f_y(x, y; x^0, y^0)\,\mathrm{d}s(x^0, y^0) + C_y, \tag{5.2b}$$

where s, $(x^0, y^0) \in [-a, a]$. In order to balance the resultant force on the crack line, Eqs. (5.2a) and (5.2b) should be set equal to the resultant forces, $F_x^0(x, y)$ and $F_y^0(x, y)$ corresponding to the externally applied tractions

$$\begin{aligned} 2\,\mathrm{Real}\Bigg[\int_{-a}^{a} \Big\{ &\left(A_1 s_1^2 \log(z_1 - s) + A_2 s_2^2 \log(z_2 - s)\right) b_x(s) - (A_1 s_1 \log(z_1 - s) \\ &+ A_2 s_2 \log(z_2 - s)) b_y(s) \Big\} \mathrm{d}s \Bigg] = F_x^0(x, y) + C_x \end{aligned} \tag{5.3a}$$

$$\begin{aligned} -2\,\mathrm{Real}\Bigg[\int_{-a}^{a} \{ &(A_1 s_1 \log(z_1 - s) + A_2 s_2 \log(z_2 - s)) b_x(s) - (A_1 \log(z_1 - s) \\ &+ A_2 \log(z_2 - s)) b_y(s) \} \mathrm{d}s \Bigg] = F_y^0(x, y) + C_y. \end{aligned} \tag{5.3b}$$

These are typical 'coupled singular integral equations', to be solved for the unknown functions $b_x(s)$ and $b_y(s)$. They can be solved numerically, subject to the auxiliary conditions, Eqs. (C3a) and (C2).

The integral equations (Eqs. (5.3a) and (5.3b)) are written with the dislocation functions as the primary unknowns. We rewrite these equations in terms of the crack-opening displacement (COD), U_x and U_y. Performing integration-by-parts and using Eqs. (C4a,b), we obtain

$$2\,\mathrm{Real}\left[\int_{-a}^{a}\left\{\left(\frac{A_1 s_1^2}{z_1-s}+\frac{A_2 s_2^2}{z_2-s}\right)U_x(s)-\left(\frac{A_1 s_1}{z_1-s}+\frac{A_2 s_2}{z_2-s}\right)U_y(s)\right\}\mathrm{d}s\right]=F_x^0(x,\,y)+C_x \tag{5.4a}$$

and

$$-2\,\mathrm{Real}\left[\int_{-a}^{a}\left\{\left(\frac{A_1 s_1}{z_1-s}+\frac{A_2 s_2}{z_2-s}\right)U_x(s)-\left(\frac{A_1}{z_1-s}+\frac{A_2}{z_2-s}\right)U_y(s)\right\}\mathrm{d}s\right]=F_y^0(x,\,y)+C_y, \tag{5.4b}$$

where $U_x(s=\pm a)=0$ and $U_y(s=\pm a)=0$ are also used.

These new integral equations, written in terms of the two components of COD, present advantages such as:

1. the 'log' functions have disappeared;
2. unlike $b_x(s)$ and $b_y(s)$, the unknowns of the problem, $U_x(s)$ and $U_y(s)$, are not singular; and
3. the consistency conditions (Eq. (C3a)) are simplified to $U_x(s=\pm a)=0$ and $U_y(s=\pm a)=0$.

These lead to a more effective numerical routine.

To illustrate the method, consider a simple example of a straight crack in a plate subjected to farfield uniform tractions which produce $F_x^0(x,\,y)=x\sigma_{xy}^{\infty}-y\sigma_{xx}^{\infty}$ and $F_y^0(x,\,y)=x\sigma_{yy}^{\infty}-y\sigma_{xy}^{\infty}$. Let the crack lie on the x-axis, so that $y=0$, which simplifies Eqs. (5.4a) and (5.4b) to

$$2\,\mathrm{Real}\left[\frac{1}{2\pi i C_{11}}\int_{-a}^{a}\left\{\left(\frac{s_1^2}{\bar{s}_1}+\frac{s_2^2}{\bar{s}_2}\right)U_x(s)-\left(\frac{s_1}{\bar{s}_1}+\frac{s_2}{\bar{s}_2}\right)U_y(s)\right\}\frac{\mathrm{d}s}{x-s}\right]=x\sigma_{xy}^{\infty}, \tag{5.5a}$$

$$2\,\mathrm{Real}\left[\frac{-1}{2\pi i C_{11}}\int_{-a}^{a}\left\{\left(\frac{s_1}{\bar{s}_1}+\frac{s_2}{\bar{s}_2}\right)U_x(s)-\left(\frac{1}{\bar{s}_1}+\frac{1}{\bar{s}_2}\right)U_y(s)\right\}\frac{\mathrm{d}s}{x-s}\right]=x\sigma_{yy}^{\infty}. \tag{5.5b}$$

A numerical routine can be set up for this set of coupled Cauchy-type-singular integral equations. However, this simple case can be solved analytically (see Eq. (B1e)). The analytical results for U_x and U_y are identical to those obtained in Section 4.1 (see Eqs. (4.1.14a) and (4.1.14b)). In brief, it is expected that Eqs. (5.3a) and (5.3b) leads to more accuracy when compared with the traction method boundary conditions. Moreover, Eqs. (5.4a) and (5.4b) are easier than Eqs. (5.3a) and (5.3b) to be implemented into a numerical routine.

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Appendix A. The characteristic equation and related formulae

For the orthotropic materials, $C_{16}=C_{26}=0$. The characteristics equation Eq. (2.3) then reduces to

$$\hat{C}_{11}\hat{s}^4+\left(2\hat{C}_{12}+\hat{C}_{66}\right)\hat{s}^2+\hat{C}_{22}=0, \tag{A1}$$

where the superposed caret, ^, denotes components with respect to the x_1, x_2-coordinate system of Fig. 1. The on-axis orthotropic constants can be written in terms of the engineering material constants, Young's moduli, shear modulus and Poissons ratios, as

$$\hat{C}_{11}=\frac{1}{E_{11}},\quad \hat{C}_{22}=\frac{1}{E_{22}},\quad \hat{C}_{66}=\frac{1}{E_{66}}=\frac{1}{G_{12}},\quad \hat{C}_{12}=\frac{-\nu_{12}}{E_{11}}=\frac{-\nu_{21}}{E_{22}}; \tag{A2}$$

note that for isotropic materials, $E_{11}=E_{22}=E$, $E_{66}=G$ and $\nu_{12}=\nu_{21}=\nu$, where $E=2G(1+\nu)$. Now, Eq. (A1) becomes $\hat{s}^4+2q\hat{s}^2+p^2=0$, where p and q are

$$p^2=\frac{\hat{C}_{22}}{\hat{C}_{11}}=\frac{E_{11}}{E_{22}},\quad q=\frac{2\hat{C}_{12}+\hat{C}_{66}}{2\hat{C}_{11}}=\frac{E_{11}}{2E_{66}}-\nu_{12}. \tag{A3}$$

The four roots of this quadric equation are $\hat{s}_1=\alpha_1+i\beta_1$, $\hat{s}_2=\alpha_2+i\beta_2$, $\hat{s}_3=\overline{\hat{s}_1}$ and $\hat{s}_4=\overline{\hat{s}_2}$, where

$$\alpha_1=\frac{\sqrt{p-q}}{\sqrt{2}},\quad \alpha_2=-\frac{\sqrt{p-q}}{\sqrt{2}},\quad \beta_1=\beta_2=\frac{\sqrt{p+q}}{\sqrt{2}}\quad \text{for } p\geq q; \tag{A4a}$$

$$\alpha_1=\alpha_2=0,\quad \beta_1=\frac{\sqrt{q+p}+\sqrt{q-p}}{\sqrt{2}},\quad \beta_2=\frac{\sqrt{q+p}-\sqrt{q-p}}{\sqrt{2}},\quad \text{for } p\geq q. \tag{A4b}$$

These equations give $\hat{s}_1$ and $\hat{s}_2$ in terms of the engineering constants explicitly. It is noteworthy that for this class of materials (orthotropic), and with respect to the x_1 x_2-coordinate system, the roots of the characteristic equation are either $\hat{s}_1=\alpha_0+i\beta_0$ and $\hat{s}_2=-\alpha_0+i\beta_0$ or $\hat{s}_1=i\beta_1$ and $\hat{s}_2=i\beta_2$. However, it is interesting that, for either case, the relation $\hat{s}_3\hat{s}_4-\hat{s}_1\hat{s}_2=0$ (or equivalently, $\alpha_1\beta_2+\alpha_2\beta_1=0$) is always true; this term appears in a number of formulae, e.g. (Eqs. (A5b) and (A5f)). Once $\hat{s}_1$ and $\hat{s}_2$ are calculated by Eqs. (A4a,b), we use the transformation formula (2.9b) to obtain s_1 and s_2 in other coordinate systems (e.g. the x, y-coordinate system of Fig. 1, with $\omega=-\theta$).

The following expressions are used in various equations in this paper:

$$\frac{1}{\tilde{s}_1}+\frac{1}{\tilde{s}_2}=\frac{1}{\tilde{s}_3}-\frac{1}{\tilde{s}_4}=\frac{s_1+s_2-s_3-s_4}{(s_1-s_3)(s_1-s_4)(s_2-s_3)(s_2-s_4)}=\frac{i\Delta_1}{\Delta_0}, \tag{A5a}$$

$$\frac{s_1}{\tilde{s}_1}+\frac{s_2}{\tilde{s}_2}=-\frac{s_3}{\tilde{s}_3}-\frac{s_4}{\tilde{s}_4}=\frac{s_3s_4-s_1s_2}{(s_1-s_3)(s_1-s_4)(s_2-s_3)(s_2-s_4)}=\frac{i\Delta_2}{\Delta_0} \tag{A5b}$$

and

$$\frac{s_1^2}{\tilde{s}_1}+\frac{s_2^2}{\tilde{s}_2}=-\frac{s_3^2}{\tilde{s}_3}-\frac{s_4^2}{\tilde{s}_4}=\frac{s_1s_3(s_4-s_2)+s_2s_4(s_3-s_1)}{(s_1-s_3)(s_1-s_4)(s_2-s_3)(s_2-s_4)}=\frac{i\Delta_3}{\Delta_0}, \tag{A5c}$$

where

$$\Delta_0=(s_1-s_3)(s_1-s_4)(s_2-s_3)(s_2-s_4)=4\beta_1\beta_2\left[(\alpha_1-\alpha_2)^2+(\beta_1+\beta_2)^2\right]>0, \tag{A5d}$$

$$\Delta_1 = -i(s_1 + s_2 - s_3 - s_4) = 2(\beta_1 + \beta_2) > 0, \tag{A5e}$$

$$\Delta_2 = -i(s_3 s_4 - s_1 s_2) = -2(\alpha_1 \beta_2 + \alpha_2 \beta_1), \tag{A5f}$$

and

$$\Delta_3 = -i[s_1 s_3 (s_4 - s_2) + s_2 s_4 (s_3 - s_1)] = -2\left[\beta_1(\alpha_2^2 + \beta_2^2) + \beta_2(\alpha_1^2 + \beta_1^2)\right] < 0; \tag{A5g}$$

$$\Delta_0 = -(\Delta_1 \Delta_3 + \Delta_2 \Delta_2) > 0, \quad \Delta_3 = s_1 s_2 \Delta_1 + (s_1 + s_2)\Delta_2 < 0, \tag{A6a}$$

$$C_{22} = s_1 s_2 s_3 s_4 C_{11} = \left(\alpha_1^2 + \beta_1^2\right)\left(\alpha_2^2 + \beta_2^2\right) C_{11}, \tag{A6b}$$

$$2C_{16} = (s_1 + s_2 + s_3 + s_4) C_{11} = 2(\alpha_1 + \alpha_2) C_{11}, \tag{A6c}$$

$$2C_{26} = [s_1 s_2 (s_3 + s_4) + s_3 s_4 (s_1 + s_2)] C_{11} = 2\left[\alpha_1\left(\alpha_2^2 + \beta_2^2\right) + \alpha_2\left(\alpha_1^2 + \beta_1^2\right)\right] C_{11}, \tag{A6d}$$

$$2C_{12} + C_{66} = [s_1(s_2 + s_3 + s_4) + s_2(s_3 + s_4) + s_3 s_4] C_{11} = \left[\alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2 + 4\alpha_1 \alpha_2\right] C_{11}, \tag{A6e}$$

$$\alpha + \bar{\alpha} = -2\frac{\Delta_2}{\Delta_3}, \qquad \alpha - \bar{\alpha} = \pm 2i\frac{\sqrt{\Delta_0}}{\Delta_3}, \qquad \alpha\bar{\alpha} = -\frac{\Delta_1}{\Delta_3} \tag{A6f}$$

Parameters α and C_0 defined in Section 3 are, in general, some functions of the material constants. However, for the special case of orthotropy ($\Delta_2 = 0$), these parameters simplify to

$$\alpha^2 = \frac{1}{s_1 s_2}, \qquad C_0 = \alpha \frac{C_{12}/C_{11} - s_1 s_2}{s_1 + s_2}, \tag{A7}$$

where, for both orthotropic cases reported in Eqs. (A4a,b),

$$s_1 s_2 = -p = -\sqrt{\frac{C_{22}}{C_{11}}}, \qquad s_1 + s_2 = i\sqrt{2(p+q)} = i\sqrt{2\left(\sqrt{\frac{C_{22}}{C_{11}}} + \frac{2C_{12} + C_{66}}{2C_{11}}\right)}. \tag{A8}$$

Thus, for orthotropic media, Eqs. (A7), (A8) show that α is purely imaginary and C_0 is a real number; these facts are helpful for simplifying the formulae of Section 3 from the anisotropic case to orthotropic case. For plane stress, the C_{ij}'s are given by Eq. (A2) and thus, α and C_0 can be easily written in terms of the on-axis orthotropic material constants. For plane strain, the C_{ij}'s of Eq. (A2) should be replaced by $C_{ij} - C_{i3}C_{j3}/C_{33}$. Nevertheless, for the case of isotropy, Eq. (A7) gives $\alpha = i$, and results in $C_0 = (1-\nu)/2$ and $C_0 = (1-2\nu)/(1-\nu)/2$ for the plane-stress and plane-strain conditions, respectively.

Appendix B. The solution of relevant integrals

The integral equation (Hilbert problem),

$$A \int_{-a}^{a} \frac{F(t)}{x-t}\,\mathrm{d}t + P(x) = 0, \ -a \le x \le a, \tag{B1a}$$

has the following solution (Muskhelishvili, 1953):

$$F(x) = \frac{1}{\pi^2 A} \frac{1}{\sqrt{a^2 - x^2}} \int_{-a}^{a} \frac{\sqrt{a^2-t^2}}{x-t} P(t)\mathrm{d}t + \frac{C_0}{\pi\sqrt{a^2-x^2}}, \tag{B1b}$$

where

$$C_0 = \int_{-a}^{a} F(t)\,\mathrm{d}x \tag{B1c}$$

serves as an auxiliary condition to render the solution unique. In particular, when $P(x) = P_0 = constant$ and $C_0 = 0$, then

$$F(x) = \frac{P_0}{\pi A} \frac{x}{\sqrt{a^2-x^2}}. \tag{B1d}$$

For the case when the unknown function $F(x)$ is bounded at both ends ($x = \pm a$), the solution of Eq. (B1a) is:

$$F(x) = \frac{\sqrt{a^2-x^2}}{\pi^2 A} \int_{-a}^{a} \frac{P(t)}{(x-t)\sqrt{a^2-t^2}}\,\mathrm{d}t, \tag{B1e}$$

provided that

$$\int_{-a}^{a} \frac{P(x)}{\sqrt{a^2-x^2}}\,\mathrm{d}x = 0. \tag{B1f}$$

The following definite integrals are valid for $w \notin [-a, +a]$, $w_0 \notin [-a, +a]$, $b \in [-a, +a]$, and $x \in [-a, +a]$:

$$\int_{-a}^{a} \frac{1}{(w \pm t)\sqrt{a^2-t^2}}\,\mathrm{d}t = \frac{\pi}{\sqrt{w^2-a^2}}, \tag{B2a}$$

$$\int_{-a}^{a} \frac{\sqrt{a^2-t^2}}{w-t}\,\mathrm{d}t = \pi\left(\sqrt{w^2-a^2} - w\right), \tag{B2b}$$

$$\int_{-a}^{a} \frac{t}{(w-t)\sqrt{a^2-t^2}}\,\mathrm{d}t = -\pi\left\{1 - \frac{w}{\sqrt{w^2-a^2}}\right\}, \tag{B2c}$$

$$\int_{-a}^{a} \frac{1}{(w-t)(t \pm w_0)\sqrt{a^2-t^2}}\,\mathrm{d}t = \frac{\pi}{w \pm w_0}\left\{\frac{1}{\sqrt{w^2-a^2}} \pm \frac{1}{\sqrt{w_0^2-a^2}}\right\}, \tag{B2d}$$

$$\int_{-a}^{a} \frac{\sqrt{a^2-t^2}}{(t-w)(t-w_0)}\,\mathrm{d}t = \frac{\pi}{w-w_0}\left\{\sqrt{w^2-a^2} - \sqrt{w_0^2-a^2} - (w-w_0)\right\}, \tag{B2e}$$

$$\int_{-a}^{a} \frac{1}{(w-t)(t-b)\sqrt{a^2-t^2}}\,\mathrm{d}t = \frac{\pi}{w-b}\left\{\frac{1}{\sqrt{w^2-a^2}}\right\}, \tag{B2f}$$

$$\int_{-a}^{a} \frac{\sqrt{a^2-t^2}}{x-t}\frac{1}{t\mp w}\,\mathrm{d}t = \pi\left(1\pm\frac{\sqrt{w^2-a^2}}{x\mp w}\right) \tag{B2g}$$

and

$$\int_{-a}^{b} \frac{\sqrt{(a+t)(b-t)}}{t-x}\,\mathrm{d}t = -\pi x + \frac{b-a}{2}\pi \quad \text{for} \quad -a<x<b \tag{B3a}$$

and

$$\int_{-a}^{b} \frac{\sqrt{(a+t)(b-t)}}{t-x}\,\mathrm{d}t = -\pi x + \frac{b-a}{2}\pi + \pi\sqrt{(a+x)(x-b)} \quad \text{for} \quad x>b. \tag{B3b}$$

Appendix C. The solution method for a relevant system of singular integral equations

Some of the problems discussed in this paper involve a system of coupled singular integral equations of the form

$$\int_{-1}^{1} \frac{M_{i1}(t,s)b_x(t)+M_{i2}(t,s)b_y(t)}{t-s}\,\mathrm{d}t + \int_{-1}^{1}[K_{i1}(t,s)b_x(t)+K_{i2}(t,s)b_y(t)]\mathrm{d}t = P_i(s), \tag{C1}$$

where $i = 1, 2$ and the coordinate along the line of discontinuity (e.g. $[-a, a]$ for an initial crack and $[0, L]$ for a kink) is normalized to $t=[-1, 1]$ and $s=[-1, 1]$. In Eq. (C1), the functions M_{ij} and the input functions P_i are known, and the kernels K_{ij} are also known and bounded in $t=[-1, 1]$. For the dislocation density functions, $b_x(t)$ and $b_y(t)$, a general form of

$$b_x(t) = \frac{B_x(t)}{\sqrt{1-t^2}}, \quad b_y(t) = \frac{B_y(t)}{\sqrt{1-t^2}} \tag{C2}$$

is assumed, where the unknown functions $B_x(t)$ and $B_y(t)$ are continuous and smooth. To render the solution of Eq. (C1) unique, the consistency (or auxiliary) conditions

$$\int_{-1}^{1} b_x(t)\mathrm{d}t = 0, \quad \int_{-1}^{1} b_y(t)\mathrm{d}t = 0, \tag{C3a}$$

$$B_x(t=-1) = 0, \quad B_y(t=-1) = 0, \tag{C3b}$$

are used, where Eq. (C3a) is for the case of an existing internal crack, and Eq. (C3b) is for a kink with its knee at $t=-1$. The first two conditions ensure that crack is closed at both ends and the other two compensate for the fact that the singularity at the kink knee is less than one half (see Bogy, 1971; Obata et al., 1989; Azhdari, 1995).

Except for special cases (see Eq. (B1a) and also Sections 4 and 5), the pair of singular integral equations, Eq. (C1), must be solved numerically. As an example, one numerical method by Gerasoulis (1982), is briefly explained below; for alternative methods, see the references cited therein. First, $B_x(t)$ and $B_y(t)$ are interpolated using N piece-wise quadratic polynomials; this creates $(2N + 1)$ nodal unknowns for B_x and likewise for B_y. Then, the singular parts of the integrals are integrated analytically, and the non-singular parts are obtained numerically. Eq. (C1) is satisfied at $2(2N)$ collocation points, each one of which is chosen to be in the middle of every two consecutive nodal points; this generates $2(2N)$ equations. Taking into consideration the auxilliary conditions (Eqs. (C3a) and (C3b)), a system of $2(2N + 1)$ algebraic linear equations is obtained. Once this system is solved, values of B_x and B_y, at $2N + 1$ distinct points along the crack (or kink) line, are obtained. Finally, considering the following relations between dislocation density functions and the CODs:

$$U_x(\zeta) = U_x(\zeta_0) - \int_{\zeta_0}^{\zeta} b_x(t)\,\mathrm{d}t, \qquad b_x(t) = -\frac{\mathrm{d}U_x(t)}{\mathrm{d}t}, \tag{C4a}$$

$$U_y(\zeta) = U_y(\zeta_0) - \int_{\zeta_0}^{\zeta} b_y(t)\mathrm{d}t, \qquad b_x(t) = -\frac{\mathrm{d}U_x(t)}{\mathrm{d}t}, \tag{C4b}$$

the COD at any point along the crack or kink line, is calculated.

Appendix D. Near-tip (asymptotic) fields, stress intensity factors (SIFs) and fracture critera

D.1. Conventional SIFs, the asymptotic stresses and displacements near the crack tip

Consider a crack (curved or straight) in a plate and assume that the roots of the characteristic equation are calculated with respect to the body coordinate systems, x–y. The conventional stress intensity factors, K_{I} and K_{II}, at the tip of this crack are defined as:

$$K_{\mathrm{I}} = \lim_{\zeta^+ \to 0}\left[\sqrt{2\pi\zeta}\sigma_{\eta\eta}\right], \quad K_{\mathrm{II}} = \lim_{\zeta^+ \to 0}\left[\sqrt{2\pi\zeta}\sigma_{\zeta\eta}\right], \tag{D1}$$

where the ζ–η coordinate system is attached to the crack-tip and the ζ-axis is tangent to the crack at its tip, making an angle ω_{t} ('t' stands for tip) with the x-axis. Now, consider a point near the crack tip (the coordinates of this point with respect to the ζ–η coordinate system are r and θ; the distance r is assumed to be very small in comparison with the crack length). The stresses and displacements at this point in terms of K_{I} and K_{II} are

$$\sigma_{\zeta\zeta} = \frac{1}{\sqrt{2\pi r}}\,\mathrm{Real}\left[\frac{1}{\hat{s}_1 - \hat{s}_2}\left\{\frac{-\hat{s}_1^2(\hat{s}_2 K_{\mathrm{I}} + K_{\mathrm{II}})}{\sqrt{\cos\theta + \hat{s}_1 \sin\theta}} + \frac{\hat{s}_2^2(\hat{s}_1 K_{\mathrm{I}} + K_{\mathrm{II}})}{\sqrt{\cos\theta + \hat{s}_2 \sin\theta}}\right\}\right], \tag{D2a}$$

$$\sigma_{\eta\eta} = \frac{1}{\sqrt{2\pi r}}\,\mathrm{Real}\left[\frac{1}{\hat{s}_1 - \hat{s}_2}\left\{\frac{-\hat{s}_2 K_{\mathrm{I}} - K_{\mathrm{II}}}{\sqrt{\cos\theta + \hat{s}_1 \sin\theta}} + \frac{\hat{s}_1 K_{\mathrm{I}} + K_{\mathrm{II}}}{\sqrt{\cos\theta + \hat{s}_2 \sin\theta}}\right\}\right], \tag{D2b}$$

$$\sigma_{\zeta\eta} = \frac{1}{\sqrt{2\pi r}}\,\mathrm{Real}\left[\frac{1}{\hat{s}_1 - \hat{s}_2}\left\{\frac{\hat{s}_1(\hat{s}_2 K_{\mathrm{I}} + K_{\mathrm{II}})}{\sqrt{\cos\theta + \hat{s}_1 \sin\theta}} + \frac{-\hat{s}_2(\hat{s}_1 K_{\mathrm{I}} + K_{\mathrm{II}})}{\sqrt{\cos\theta + \hat{s}_2 \sin\theta}}\right\}\right], \tag{D2c}$$

$$u_{\zeta} = \sqrt{\frac{2r}{\pi}}\,\mathrm{Real}\left[\frac{1}{\hat{s}_1 - \hat{s}_2}\left\{-\hat{p}_1(\hat{s}_2 K_{\mathrm{I}} + K_{\mathrm{II}})\sqrt{\cos\theta + \hat{s}_1 \sin\theta} + \hat{p}_2(\hat{s}_1 K_{\mathrm{I}} + K_{\mathrm{II}})\sqrt{\cos\theta + \hat{s}_2 \sin\theta}\right\}\right] \tag{D3a}$$

and

$$u_\eta = \sqrt{\frac{2r}{\pi}}\,\mathrm{Real}\left[\frac{1}{\hat{s}_1 - \hat{s}_2}\left\{ -\hat{q}_1(\hat{s}_2 K_\mathrm{I} + K_\mathrm{II})\sqrt{\cos\theta + \hat{s}_1 \sin\theta} + \hat{q}_2(\hat{s}_1 K_\mathrm{I} + K_\mathrm{II})\sqrt{\cos\theta + \hat{s}_2 \sin\theta}\right\}\right], \quad \text{(D3b)}$$

where $r \equiv \sqrt{\zeta^2 + \eta^2}$, $\theta \equiv \arctan(\eta/\zeta)$ and the parameters $\hat{s}_j$, $\hat{p}_j$ and $\hat{q}_j$ (j = 1, 2) are obtained from s_j, p_j and q_j through a rotation by the angle ω_t, see Eq. (2.9b). These expressions are valid for the points in a small vicinity of the crack tip.

A more useful form for the displacement fields is given in terms of the crack-opening displacements (COD). From Eqs. (D3a) and (D3b), the CODs are

$$[u_\zeta] = \sqrt{\frac{2r}{\pi}}\,2\,\mathrm{Real}\left[\frac{i}{\hat{s}_1 - \hat{s}_2}\left\{K_\mathrm{I}\left(\hat{s}_1\hat{p}_2 - \hat{s}_2\hat{p}_1\right) + K_\mathrm{II}\left(\hat{p}_2 - \hat{p}_1\right)\right\}\right] \quad \text{(D4a)}$$

and

$$[u_\eta] = \sqrt{\frac{2r}{\pi}}\,2\,\mathrm{Real}\left[\frac{i}{\hat{s}_1 - \hat{s}_2}\left\{K_\mathrm{I}\left(\hat{s}_1\hat{q}_2 - \hat{s}_2\hat{q}_1\right) + K_\mathrm{II}\left(\hat{q}_2 - \hat{q}_1\right)\right\}\right], \quad \text{(D4b)}$$

where r is very small and denotes a point along the crack line (r is on the negative side of the ζ-axis). If the boundary-value problem (BVP) of a kinked crack is solved via a combination of analytical/numerical/asymptotic methods, then K_I and K_II are usually known: the use of Eqs. (D2a), (D2b), (D2c), (D3a), (D3b), (D4a) and (D4b) gives the stresses and displacements in the crack-tip vicinity. On the other hand, if the BVP is solved by, e.g., finite-element method, then usually CODs are known; in this case, Eqs. (D4a) and (D4b) gives the K_I and K_II. For this, CODs at a sufficiently small r should be used in (D4a,b).

If a BVP is solved by the method of 'distribution of edge-dislocations along the crack line', then naturally, the edge-dislocation density function along the crack line is calculated and thus known; see Appendix C. Denote the x- and y-components of the regular part of the edge-dislocation density at the crack tip by $B_x(s = L)$ and $B_y(s = L)$, where s measures length along the crack line and L is the crack or the kink length. Assuming that s = 0 is where the left crack-tip or the kink-knee is located, then K_I and K_II at the right crack tip can be calculated by (Obata et al., 1989)

$$K_\mathrm{I} = \pi\sqrt{\frac{2\pi}{L}}[H_{11}B_x(s = L) + H_{12}B_y(s = L)], \quad K_\mathrm{II} = \pi\sqrt{\frac{2\pi}{L}}[H_{21}B_x(s = L) + H_{22}B_y(s = L)], \quad \text{(D5a)}$$

where

$$H_{11} = 2\,\mathrm{Real}\left[\frac{1}{2\pi i C_{11}}\left\{\frac{s_1}{\tilde{s}_1}F_1 + \frac{s_2}{\tilde{s}_2}F_2\right\}\right], \quad H_{12} = 2\,\mathrm{Real}\left[\frac{-1}{2\pi i C_{11}}\left\{\frac{1}{\tilde{s}_1}F_1 + \frac{1}{\tilde{s}_2}F_2\right\}\right], \quad \text{(D5b)}$$

$$H_{21} = 2\,\mathrm{Real}\left[\frac{1}{2\pi i C_{11}}\left\{\frac{s_1}{\tilde{s}_1}G_1 + \frac{s_2}{\tilde{s}_2}G_2\right\}\right], \quad H_{22} = 2\,\mathrm{Real}\left[\frac{-1}{2\pi i C_{11}}\left\{\frac{1}{\tilde{s}_1}G_1 + \frac{1}{\tilde{s}_2}G_2\right\}\right], \quad \text{(D5c)}$$

where

$$F_j = \cos\omega_\mathrm{t} + s_j \sin\omega_\mathrm{t}, \quad G_j = \sin\omega_\mathrm{t} - s_j\cos\omega_\mathrm{t}. \quad \text{(D5d)}$$

D.2. Hoop and shear stress intensity factors around the crack tip

Once K_{I} and K_{II} are calculated at a crack or kink tip, the stress field in the vicinity of the tip is known and, thus, the hoop stress intensity factor (HSIF or $K_{\theta\theta}$) and the shear stress intensity factor (SSIF or $K_{r\theta}$) can be calculated easily (Azhdari, 1995). The following are the formulae for HSIF and SSIF at angle θ measured with respect to the ζ-axis:

$$K_{\theta\theta} = K_{11}K_{\mathrm{I}} + K_{12}K_{\mathrm{II}}, \quad K_{r\theta} = K_{21}K_{\mathrm{I}} + K_{22}K_{\mathrm{II}}, \tag{D6a}$$

where

$$K_{11} = \mathrm{Real}\left[\frac{1}{\hat{s}_2 - \hat{s}_1}\left\{\hat{s}_2(c+\hat{s}_1 s)^{3/2} - \hat{s}_1(c+\hat{s}_2 s)^{3/2}\right\}\right], \tag{D6b}$$

$$K_{12} = \mathrm{Real}\left[\frac{1}{\hat{s}_2 - \hat{s}_1}\left\{(c+\hat{s}_1 s)^{3/2} - (c+\hat{s}_2 s)^{3/2}\right\}\right], \tag{D6c}$$

$$K_{21} = \mathrm{Real}\left[\frac{1}{\hat{s}_2 - \hat{s}_1}\left\{\hat{s}_2(c+\hat{s}_1 s)^{1/2}(s-\hat{s}_1 c) - \hat{s}_1(c+\hat{s}_2 s)^{1/2}(s-\hat{s}_2 c)\right\}\right] \tag{D6d}$$

and

$$K_{22} = \mathrm{Real}\left[\frac{1}{\hat{s}_2 - \hat{s}_1}\left\{(c+\hat{s}_1 s)^{1/2}(s-\hat{s}_1 c) - (c+\hat{s}_2 s)^{1/2}(s-\hat{s}_2 c)\right\}\right], \tag{D6e}$$

where $s = \sin\theta$, $c = \cos\theta$ and $\hat{s}_i$ is the transformed form of s_i from the x–y to the ζ–η coordinate system attached at the tip (see Eq. (2.9b)). Note that, angle θ is measured with respect to the ζ-axis ($-\pi \leq \theta \leq \pi$) and, thus, this angle with respect to the x-axis is $\omega = \omega_{\mathrm{t}} + \theta$.

D.3. Fracture criteria: max-hoop stress intensity factor and max-Mode-I SIF

As can be seen from Eqs. (D6a–e), the two equations, $\partial K_{\theta\theta}/\partial\theta = 0$ and $K_{r\theta} = 0$, are identical. Thus, the critical angle $\theta = \theta_{\mathrm{c}}$, at which $K_{\theta\theta}$ is maximum, renders $K_{r\theta}$ zero. This is the basis for the max-HSIF (or equivalently, zero-SSIF) fracture criterion. Accordingly, crack (or a kink) may propagate in the direction for which $K_{\theta\theta}$ is maximum (see Azhdari and Nemat-Nasser, 1996a and 1998).

More appropriate is the max-K_{I} fracture criterion. Consider a crack and the corresponding coordinate systems x–y and ζ–η, as defined above. Assume, now, a vanishingly small kink of length l at the tip of this crack, making an angle γ, measured with respect to the ζ-axis. This new configuration (crack plus the kink) creates a new BVP. After solving this BVP (almost always numerically), the corresponding K_{I} and K_{II} at the tip of the kink can be calculated via, e.g. (D4 or D5); we denote these SIFs by $K_{\mathrm{I}}^{(\mathrm{k})}$ and $K_{\mathrm{II}}^{(\mathrm{k})}$, where superscript '(k)' stands for kink. The conventional max-K_{I} (or equivalently, zero-K_{II}) fracture criterion states that a pre-existing crack propagates in the critical direction of $\gamma = \gamma_{\mathrm{c}}$, for which $K_{\mathrm{I}}^{(\mathrm{k})}$ is a maximum (or $K_{\mathrm{II}}^{(\mathrm{k})} = 0$). It is interesting to note that the values of $K_{\mathrm{I}}^{(\mathrm{k})} = 0$ and $K_{\mathrm{II}}^{(\mathrm{k})} = 0$ are independent of the kink length l when it is vanishingly small (see Azhdari and Nemat-Nasser, 1996b). Note that the determination of the critical kinking-angle by the $\max - K_{\mathrm{I}}^{(\mathrm{k})}$ is computationally much more expensive than the one computed via the max-HSIF criterion.

The max-HSIF and $\max - K_{\mathrm{I}}^{(\mathrm{k})}$ fracture criteria do not, in general, predict the same propagation path for a given problem (see Azhdari and Nemat-Nasser, 1996a). However, for various combinations of relevant parameters (for different material properties, material symmetry orientation and loading), and for small kink angles (within the range of ± 8 degrees), the magnitude of the SIFs predicted by these two fracture criteria are less than 1% apart; this holds for much larger kink angles when the material is isotropic. Moreover, Azhdari and Nemat-Nasser (1996b) showed that the kink-direction predictions made by the maximum energy-release rate criterion, in general, do not accord with either the ones predicted by the max-HSIF or by the $\max - K_{\rm I}^{(\rm k)}$ criteria.

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