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Bridged interface cracks in anisotropic bimaterials

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ABSTRACT

The bonding strength between dissimilar materials in composites is often relatively weak. Interface debonding and cracking are of common occurrence. As an effective method to minimize such failures, fibre stitching is used to enhance the bonding strength when joining layers in polymer composites. With the presence of the reinforcing fibre, the overall elastic response for each component of the composite is, in general, anisotropic. This paper investigates the elastic behaviour of bridged interface cracks in anisotropic bimaterials. From the equilibrium of the interface crack, a system of Cauchy integral equations for the required distributed dislocation density is obtained. When the bridging force depends linearly on the crack-opening displacement, explicit solutions are given in terms of a series of Jacobi polynomials. For illustration, a bridged interface crack in isotropic bimaterials is examined in detail. Results suggest that bridging fibres effectively enhance the toughness of the composite. When the bridging force increases, the effect of interface mismatch becomes insignificant, being overshadowed by the effect of the bridging forces.

§1. INTRODUCTION

Different materials are combined to design composites with improved overall properties. Since the bonding strength between dissimilar materials is comparatively weak, interface debonding and cracking are of common occurrence. An effective method to minimize such failures in polymer composites is to use fibre stitching when joining composite layers. The stitching fibres bridge the interface microcracks and provide resistance to further fracture, therefore enhancing the bonding strength considerably. Because of the presence of the reinforcing fibres, the overall response of each component of the composite is, in general, anisotropic. A study of the elastic behaviour of bridged interface cracks in anisotropic bimaterials is necessary, but the topic does not seem to have been addressed in the literature.

Bridged cracks in homogeneous materials have been investigated by many researchers (for example Marshall *et al.* (1985), Marshall and Evans (1985), Budiansky *et al.* (1986), Horii *et al.* (1987), Nemat-Nasser and Hori (1987), Rose (1987), Swanson *et al.* (1987), Hori and Nemat-Nasser (1990), Willis and Nemat-Nasser (1990), Movchan and Willis (1993, 1996, 1998), Willis (1993)). On the other hand, problems of anisotropic elasticity, in which all field variables depend only on

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two spatial coordinates, have been examined by Eshelby *et al.* (1953) and then developed into an elegant six-dimensional formalism by Stroh (1958, 1962). Further advances have been achieved by Malen and Lothe (1970), Barnett and Lothe (1973, 1974, 1975), Chadwick and Smith (1977) and Ting (1986, 1989, 1996). The interface crack in anisotropic bimaterials was first studied by Clements (1971) and Willis (1971), and recently by many investigators (see Ting (1996) for references).

In this paper, we examine the elastic behaviour of bridged interface cracks in anisotropic bimaterials. Based on the discussion of Nemat-Nasser and Hori (1987), the equilibrium of the forces acting over the crack surfaces produces an equation relating the bridging forces, the crack resistance and the applied tractions. The crack resistance associated with an interface crack in anisotropic bimaterials has been given by Willis (1971). Using the expression for the crack resistance, from the equilibrium equation, a system of Cauchy singular integral equations is derived. When the bridging force is linearly dependent on the crack-opening displacement, solutions in a series of Jacobi polynomials are obtained; for the numerical solution, the polynomial Galerkin method (Erdogan *et al.* 1973, and Golberg 1990) is employed. For illustration, an example for an interface bridged crack in an isotropic bimaterial is discussed in detail; results suggest that bridging fibres effectively enhance the toughness of the composite. When the bridging force increases, the effect of the interface mismatch becomes insignificant, being overshadowed by the effect of the bridging forces. Studies for nonlinear cases, that is the bridging forces depending nonlinearly on the crack-opening displacement, will be reported elsewhere.

§2. Basic equations

Let $\hat{x}_k$, $k = 1,2,3$, be a fixed Cartesian coordinate system in an anisotropic bimaterial with the interface of the bimaterial being on the $(\hat{x}_1, \hat{x}_3)$ plane. Assume that all variables are independent of $\hat{x}_3$, and a crack is located on the interface of the bimaterial extending from $\hat{x}_1 = -L$, to $\hat{x}_1 = L$, for $L > 0$.

The bridging force is assumed to be an explicit functional of the crack-opening displacement $\hat{\mathbf{u}}$, being denoted by

$$\hat{\mathbf{p}}(\hat{x}_1, \hat{\mathbf{u}}) = K_0 \hat{\mathbf{f}}(\hat{x}_1, \hat{\mathbf{u}}), \tag{1}$$

where K_0 has a physical dimension of force per unit volume, and the vector functional $\hat{\mathbf{f}}(\hat{x}_1, \hat{\mathbf{u}})$ has the dimension of length. The crack is subjected to an external traction $\hat{\mathbf{T}}(\hat{x}_1)$.

As discussed by Nemat-Nasser and Hori (1987), in view of the superposition principle, the equilibrium requires

$$\hat{\tau}(\hat{x}_1) + \hat{\mathbf{p}}(\hat{x}_1, \hat{\mathbf{u}}) + \hat{\mathbf{T}}(\hat{x}_1) = \mathbf{0}, \tag{2}$$

where $\hat{\tau}(\hat{x}_1)$, $\hat{\mathbf{p}}(\hat{x}_1, \hat{\mathbf{u}})$ and $\hat{\mathbf{T}}(\hat{x}_1)$ are the crack resistance, the bridging force and the applied tractions respectively.

The crack resistance $\hat{\tau}(x_1)$ associated with an interface crack in a general anisotropic bimaterial, is given by (Willis 1971)

$$\hat{\tau}(x_1) = \frac{1}{\pi} \mathrm{Re}\left(\frac{\Lambda}{\hat{x}_1 - \mathrm{i}0}\right) * \hat{\mathbf{B}}(\hat{x}_1), \quad \mathrm{i} = (-1)^{1/2}, \tag{3}$$

where $*$ denotes the convolution integration, and $\hat{\mathbf{B}}(\hat{x}_1)$ is the dislocation density vector. In terms of Stroh (1958, 1962) matrices, Λ is expressed as (Ni and Nemat-Nasser 1991, 1992)

$$\Lambda \equiv -\mathrm{i}[\bar{\mathbf{A}}_+\bar{\mathbf{L}}_+^{-1} - \mathbf{A}_-\mathbf{L}_-^{-1}]^{-1}, \tag{4a}$$

where the overbar denotes the complex conjugate, the subscripts $+$ and $-$ represent the material in the upper half-space ($x_2 \geqslant 0$) and lower half-space ($x_2 \leqslant 0$) respectively, and

$$\mathbf{A} \equiv [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \quad \mathbf{L} \equiv [\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3], \tag{4b}$$

where $[\mathbf{a}_j, \mathbf{l}_j]^{\mathrm{T}}$, $j = 1,2,3$, are the generalized eigenvectors, corresponding to three (maybe multiple) eigenvalues with positive imaginary parts, of the fundamental elasticity matrix $\mathbf{N}$ (Ingebrigtsen and Tonning 1969):

$$\mathbf{N} \equiv \begin{bmatrix} \mathbf{n}_{11} & \mathbf{n}_{12} \\ \mathbf{n}_{21} & \mathbf{n}_{11}^{\mathrm{T}} \end{bmatrix}, \tag{4c}$$

where

$$\mathbf{n}_{11} \equiv -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{n}_{21} \equiv \mathbf{Q} - \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{n}_{12} \equiv -\mathbf{T}^{-1}, \tag{4d}$$

and

$$\mathbf{Q} \equiv [C_{j1k1}], \qquad \mathbf{R} \equiv [C_{j1k2}], \quad \mathbf{T} \equiv [C_{j2k2}] \tag{4e}$$

Note that $\mathbf{A}$ and $\mathbf{L}$ are not yet uniquely determined, since the normalization and numbering of eigenvectors $[\mathbf{a}_j, \mathbf{l}_j]^{\mathrm{T}}$ is not specified; however, the matrix $\mathbf{A}\mathbf{L}^{-1}$ in equation (4*a*) is uniquely defined.

As far as the in-plane field is concerned, disregard the $\hat{x}_3$ components and write Λ as

$$\Lambda = \begin{bmatrix} \alpha_1 & \alpha_3 + \mathrm{i}\alpha_4 \\ \alpha_3 - \mathrm{i}\alpha_4 & \alpha_2 \end{bmatrix}, \tag{5a}$$

$$\Lambda = \Lambda_1 - \mathrm{i}\Lambda_2, \tag{5b}$$

where

$$\Lambda_1 = \begin{bmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{bmatrix} \tag{5c}$$

and

$$\Lambda_2 = \begin{bmatrix} 0 & -\alpha_4 \\ \alpha_4 & 0 \end{bmatrix}. \tag{5d}$$

The parameters α_j, $j = 1,2,3,4$, can be evaluated explicitly or numerically. As examples, in what follows, the explicit expressions for α_j for isotropic, orthotropic and monoclinic bimaterials are given in terms of the elastic constants (Suo 1990, Ni and Nemat-Nasser 1991, 1992, Ting 1992a, 1996).

(i) Isotropic bimaterial

$$\alpha_1 = \alpha_2 = C, \quad \alpha_3 = 0, \quad \alpha_4 = -C\beta_{\mathrm{D}}, \tag{6a}$$

where C and β_D, and also α_D given below, are the Dundurs (1969a,b) constants, which are defined by

$$C \equiv \frac{2\mu_+(1-\alpha_D)}{(\kappa_+ +1)(1-\beta_D^2)} = \frac{\mu_+(\kappa_- +1)+\mu_-(\kappa_+ +1)}{(\kappa_- -1)(\kappa_+ -1)}, \tag{6b}$$

$$\beta_D \equiv \frac{\mu_+(\kappa_- -1)-\mu_-(\kappa_+ -1)}{\mu_+(\kappa_- +1)+\mu_-(\kappa_+ +1)}, \tag{6c}$$

with

$$\alpha_D \equiv \frac{\mu_+(\kappa_- +1)-\mu_-(\kappa_+ +1)}{\mu_+(\kappa_- +1)+\mu_-(\kappa_+ +1)}, \tag{6d}$$

where μ and κ are the shear modulus and bulk modulus, respectively.

For the following examples ((ii) and (iii)), $[c_{ij}]$, $i,j=1,2,4,5,6$ is the elastic matrix, and the compliance matrix $[s_{ij}]$ $i,j=1,2,4,5,6$, is the inverse of the elastic matrix $[c_{ij}]$.

(ii) Orthotropic bimaterial

Assume that the x_1, x_2 and x_3 axes coincide with the material symmetry directions. Then,

$$\alpha_1 = \frac{b_1}{e}, \quad \alpha_2 = \frac{b_2}{e}, \quad \alpha_3 = 0, \quad \alpha_4 = \frac{d}{e}, \tag{7a}$$

where

$$b_1 = (r_{22})_+ + (r_{22})_-, \; b_2 = (r_{11})_+ + (r_{11})_-, \tag{7b}$$

$$d = (r_{21})_+ - (r_{21})_-, \; e = b_1 b_2 - d^2. \tag{7c}$$

In each half-space, r_{11}, r_{22} and r_{21} are given explicitly by

$$r_{11} = \frac{1}{c_{12}+c_0}\left(\frac{c_{22}(2c_0-c)}{c_{66}(c_0-c_{12})}\right)^{1/2}, \tag{7d}$$

$$r_{22} = \frac{1}{c_{12}+c_0}\left(\frac{c_{11}(2c_0-c)}{c_{66}(c_0-c_{12})}\right)^{1/2}, \tag{7e}$$

$$r_{21} = \frac{1}{c_{12}+c_0}, \tag{7f}$$

where c_0 and c are defined by

$$c_0 \equiv (c_{11}c_{22})^{1/2}, \quad c \equiv c_0 - c_{12} - 2c_{66}. \tag{7g}$$

(iii) Monoclinic bimaterial

Assume that the (x_1, x_2) plane is taken to be the only plane of symmetry of the material. Then,

$$\alpha_1 = \frac{b_1}{e}, \quad \alpha_2 = \frac{b_2}{e}, \quad \alpha_3 = -\frac{\mathrm{Im}(d)}{e}, \quad \alpha_4 = -\frac{\mathrm{Re}(d)}{e}, \tag{8a}$$

where

$$b_1 = (r_{22})_+ + (r_{22})_-, \quad b_2 = (r_{11})_+ + (r_{11})_-, \tag{8b}$$

$$d = (\bar{r}_{21})_+ - (r_{21})_-, \quad e = b_1 b_2 - |d|^2. \tag{8c}$$

In each half-space, r_{11}, r_{22} and r_{21} are defined by

$$r_{11} = s_{11}\,\mathrm{Im}(p_1 + p_2), \tag{8d}$$

$$r_{22} = s_{22}\,\mathrm{Im}\left(\frac{1}{\bar{p}_1} + \frac{1}{\bar{p}_2}\right), \tag{8e}$$

$$r_{21} = s_{12} - s_{11}p_1p_2, \tag{8f}$$

where p_1 and p_2 are the eigenvalues with positive imaginary parts of the fundamental elasticity matrix $\mathbf{N}$ and are found to be the roots of the equation

$$s_{11}p^4 - 2s_{16}p^3 + (2s_{12} + s_{66})p^2 - 2s_{26}p + s_{22} = 0. \tag{9}$$

The basic equation (2) is then rewritten as an equation for the dislocation density vector $\hat{\mathbf{B}}(\hat{x}_1)$:

$$\frac{\Lambda_1}{\pi}\int_{-L}^{L}\frac{\hat{\mathbf{B}}(\xi)}{\hat{x}_1 - \xi}\,\mathrm{d}\xi + \Lambda_2\hat{\mathbf{B}}(\hat{x}_1) + K_0\hat{\mathbf{f}}(\hat{x}_1,\hat{\mathbf{u}}) + \hat{\mathbf{T}}(\hat{x}_1) = \mathbf{0}, \tag{10a}$$

with the auxiliary condition

$$\int_{-L}^{L}\hat{\mathbf{B}}(\xi)\,\mathrm{d}\xi = \mathbf{0}. \tag{10b}$$

The crack-opening displacement $\hat{\mathbf{u}}(\hat{x}_1)$ is related to $\hat{\mathbf{B}}(x_1)$ by

$$\hat{\mathbf{u}}(x_1) = -\int_{-L}^{\hat{x}_1}\hat{\mathbf{B}}(\xi)\,\mathrm{d}\xi. \tag{10c}$$

Introduce the following dimensionless variables:

$$x_1 = \frac{\hat{x}_1}{L}, \tag{11a}$$

$$\mathbf{B}(x_1) = \hat{\mathbf{B}}(\hat{x}_1), \tag{11b}$$

$$\mathbf{T}(x_1) = \frac{\hat{\mathbf{T}}(\hat{x}_1)}{\alpha_0}, \tag{11c}$$

$$\mathbf{f}(x_1,\mathbf{u}) = \frac{\hat{\mathbf{f}}(\hat{x}_1,\hat{\mathbf{u}})}{L}, \tag{11d}$$

$$\mathbf{u}(x_1) = \frac{\hat{\mathbf{u}}(\hat{x}_1)}{L}, \tag{11e}$$

$$\Lambda_{10} = \frac{\Lambda_1}{\alpha_0}, \tag{11f}$$

$$\Lambda_{20} = \frac{\Lambda_2}{\alpha_0}, \tag{11g}$$

where $\alpha_0 = (\alpha_1\alpha_2 - \alpha_3^2)^{1/2}$ has a physical dimension of force per unit area and reduce equations (10 *a*)–(10 *c*) to the non-dimensional equations

$$\frac{\Lambda_{10}}{\pi}\int_{-1}^{1}\frac{\mathbf{B}(\xi)}{x_1-\xi}\,\mathrm{d}\xi + \Lambda_{20}\mathbf{B}(x_1) + l\hat{\mathbf{f}}(x_1,\mathbf{u}) + \mathbf{T}(x_1) = \mathbf{0}, \quad (12\,a)$$

$$\int_{-1}^{1}\mathbf{B}(\xi)\,\mathrm{d}\xi = \mathbf{0} \quad (12\,b)$$

and

$$\mathbf{u}(x_1) = -\int_{-1}^{x_1}\mathbf{B}(\xi)\,\mathrm{d}\xi. \quad (12\,c)$$

The non-dimensional quantity $l = K_0L/\alpha_0$ is defined as a dimensionless crack length, which represents the overall scale of the bridging and is proportional to the physical crack length and the strength of the bridging force; $l \ll 1$ corresponds to a 'short' crack, and $l \gg 1$ to a 'long' crack.

§ 3. Linear bridging force

3.1. *Formulation*

The following three main cases of bridging forces have been described in Nemat-Nasser and Hori (1987):

$$\hat{\mathbf{p}} = -K_0 g(x_1)\hat{\mathbf{u}}(x_1), \quad (13\,a)$$

$$\hat{\mathbf{p}} = -K_0[\hat{\mathbf{h}}(x_1) + g(x_1)\hat{\mathbf{u}}(x_1)], \quad (13\,b)$$

$$\hat{\mathbf{p}} = -K_0\mathbf{g}(x_1)(\hat{u}_1^2 + \hat{u}_2^2)^{1/2}, \quad (13\,c)$$

where K_0, again, has a physical dimension of force per unit volume, and $g(x_1)$ and $\mathbf{g}(x_1)$ are the dimensionless scalar and vector functions respectively, denoting the distribution of the bridging forces. For continuously bridged cracks, $g(x_1)$ or $\mathbf{g}(x_1)$ is continuous while, if the bridging is discrete, then $g(x_1)$ or $\mathbf{g}(x_1)$ may be represented by a Dirac delta function. As pointed out by Nemat-Nasser and Hori (1987), mathematically, cases (13 *a*) and (13 *b*) are essentially the same, since the term $K_0\hat{\mathbf{h}}(x_1)$ in equation (13 *b*) is equivalent to a change in the external applied traction.

Consider now the typical linear bridging force (13 *a*). The basic equation (12 *a*) is written as

$$\frac{\Lambda_{10}}{\pi}\int_{-1}^{1}\frac{\mathbf{B}(\xi)}{x_1-\xi}\,\mathrm{d}\xi + \Lambda_{20}\mathbf{B}(x_1) + lg(x_1)\int_{-1}^{x_1}\mathbf{B}(\xi)\,\mathrm{d}\xi + \mathbf{T}(x_1) = \mathbf{0}. \quad (14)$$

Inverting Λ_{10}, equation (14) is transformed to

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\mathbf{B}(\xi)}{x_1-\xi}\,\mathrm{d}\xi + \Lambda_{10}^{-1}\Lambda_{20}\mathbf{B}(x_1) + lg(x_1)\,\Lambda_{10}^{-1}\int_{-1}^{x_1}\mathbf{B}(\xi)\,\mathrm{d}\xi + \Lambda_{10}^{-1}\mathbf{T}(x_1) = \mathbf{0}. \quad (15)$$

As discussed by Ni and Nemat-Nasser (1992), the matrix $\Lambda_{10}^{-1}\Lambda_{20} = \Lambda_1^{-1}\Lambda_2$ can be diagonalized as

$$\mathbf{E}^{-1}[\Lambda_1^{-1}\Lambda_2]\mathbf{E} = \begin{bmatrix} -\mathrm{i}\beta_G & 0 \\ 0 & \mathrm{i}\beta_G \end{bmatrix}, \quad (16\,a)$$

where the generalized Dundurs constant β_G is defined by

$$\beta_G \equiv -\frac{\alpha_4}{\alpha_0} = \frac{-\alpha_4}{(\alpha_1\alpha_2 - \alpha_3^2)^{1/2}}, \tag{16 b}$$

and, in a dimensionless form, the matrices $\mathbf{E}$ and $\mathbf{E}^{-1}$ are explicitly given by

$$\mathbf{E} = [\mathbf{v}_1, \mathbf{v}_2] = \frac{1}{\alpha_0}\begin{bmatrix} -\alpha_3 + i\alpha_0 & -\alpha_3 - i\alpha_0 \\ \alpha_1 & \alpha_1 \end{bmatrix} \tag{16 c}$$

and

$$\mathbf{E}^{-1} = \frac{1}{2\alpha_1}\begin{bmatrix} -i\alpha_1 & -i\alpha_3 + \alpha_0 \\ i\alpha_1 & i\alpha_3 + \alpha_0 \end{bmatrix}, \tag{16 d}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors of $\Lambda_1^{-1}\Lambda_2$ corresponding to the eigenvalues $-i\beta_G$ and $i\beta_G$ respectively.

Applying $\mathbf{E}^{-1}$ to both sides of equation (15), it follows that

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\mathbf{M}(\xi)}{x_1 - \xi}\,d\xi + \begin{bmatrix} -i\beta_G & 0 \\ 0 & i\beta_G \end{bmatrix}\mathbf{M}(x_1) + lg(x_1)\,\mathbf{U}_1\int_{-1}^{x_1}\mathbf{M}(\xi)\,d\xi + \mathbf{U}_2\mathbf{T}(x_1) = \mathbf{0}, \tag{17}$$

together with the auxiliary condition

$$\int_{-1}^{1}\mathbf{M}(\xi)\,d\xi = \mathbf{0}, \tag{18 a}$$

where

$$\mathbf{M}(x_1) = \begin{bmatrix} M_1(x_1) \\ M_2(x_1) \end{bmatrix} \equiv \mathbf{E}^{-1}\mathbf{B}(x_1), \tag{18 b}$$

and the matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ are defined by

$$\mathbf{U}_1 \equiv \mathbf{E}^{-1}\Lambda_{10}^{-1}\mathbf{E} = \frac{1}{2\alpha_0\alpha_1}\begin{bmatrix} \alpha_1(a_1 + \alpha_2) & \alpha_1^2 - \alpha_0^2 + \alpha_3(\alpha_3 + 2i\alpha_0) \\ \alpha_1^2 - \alpha_0^2 + \alpha_3(\alpha_3 - 2i\alpha_0) & \alpha_1(\alpha_1 + \alpha_2) \end{bmatrix}, \tag{18 c}$$

$$\mathbf{U}_2 \equiv \mathbf{E}^{-1}\Lambda_{10}^{-1} = \frac{1}{2\alpha_1}\begin{bmatrix} -\alpha_3 - i\alpha_0 & \alpha_1 \\ -\alpha_3 + i\alpha_0 & \alpha_1 \end{bmatrix}. \tag{18 d}$$

From equations (18 *b*) it is seen that

$$M_2(x_1) = \overline{M_1(x_1)}. \tag{19}$$

Hence, in the system (17), the two coupled equations are each other's complex conjugate. These, in general, may not be decoupled, except for some special cases; see the example below.

3.2. *Solution method*

The system (17) of coupled Cauchy singular integral equations of the second kind is solved in terms of series of Jacobi polynomials. Express $\mathbf{M}(x_1)$ in the following form:

$$\mathbf{M}(x_1) = \begin{bmatrix} M_1(x_1) \\ M_2(x_1) \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} a_n P_n^{(\alpha,\beta)}(x_1) w(x_1) \\ \bar{a}_n P_n^{(\beta,\alpha)}(x_1) \overline{w(x_1)} \end{bmatrix}, \tag{20 a}$$

where $P_n^{(\alpha,\beta)}(x_1)$ are the Jacobi polynomials defined by

$$P_n^{(\alpha,\beta)}(x_1) = \frac{1}{w(x_1)} \frac{(-1)^n}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} [w(x_1)(1-x_1^2)^n], \tag{20 b}$$

for $n = 0, 1, 2, \ldots$, and the weight function is given by

$$w(x_1) = (1-x_1)^{\alpha}(1+x_1)^{\beta}. \tag{20 c}$$

The parameters α and β are defined by

$$\alpha = -\tfrac{1}{2} - \mathrm{i}\gamma_0, \tag{20 d}$$

$$\beta = -\tfrac{1}{2} + \mathrm{i}\gamma_0, \tag{20 e}$$

with

$$\gamma_0 = \frac{1}{2\pi} \ln \left| \frac{1-\beta_{\mathrm{G}}}{1+\beta_{\mathrm{G}}} \right|. \tag{20 f}$$

Apply condition (18 *a*), and use the orthogonality of the Jacobi polynomials (see below), to show that $a_0 = 0$. Thus, the sum in equation (20 *a*) actually starts from $n = 1$.

Employing the Galerkin method (Erdogan *et al.* 1973, Mikhlin and Prössdorf 1986, Goldberg 1990), approximate $\mathbf{M}(x_1)$ by a finite series:

$$\mathbf{M}(x_1) = \begin{bmatrix} M_1(x_1) \\ M_2(x_1) \end{bmatrix} \approx \sum_{n=1}^{N} \begin{bmatrix} a_n P_n^{(\alpha,\beta)}(x_1) w(x_1) \\ \bar{a}_n P_n^{(\alpha,\beta)}(x_1) \overline{w(x_1)} \end{bmatrix}. \tag{21}$$

Then, substitute this approximation into equation (17) and reduce it to the following equation for complex unknowns a_n, $n = 1, 2, \ldots, N$:

$$\sum_{n=1}^{N} \left\{ a_n[(1-\beta_{\mathrm{G}}^2)^{1/2} P_{n-1}^{(-\alpha,-\beta)}(x_1) + \frac{\alpha_1+\alpha_2}{2n\alpha_0}\, lg(x_1)\,(1-x_1)^{-\beta}(1+x_1)^{-\alpha} P_{n-1}^{(-\beta,-\alpha)}(x_1)] \right.$$
$$\left. + \bar{a}_n \left[\left(\frac{\alpha_1^2 + 2\alpha_3^2 - \alpha_1\alpha_2 + 2\mathrm{i}\alpha_3\alpha_0}{2n\alpha_0\alpha_1} \right) lg(x_1)\,(1-x_1)^{-\alpha}(1+x_1)^{-\beta} P_{n-1}^{(-\alpha,-\beta)}(x_1) \right] \right\}$$
$$= -\left(\frac{\alpha_3 + \mathrm{i}\alpha_0}{\alpha_1} T_1(x_1) - T_2(x_1) \right). \tag{22}$$

In the process of deriving equation (22), we have made use of the following two identities of Jacobi polynomials (for example Erdelyi *et al.* (1953)):

$$\frac{1}{\pi} \int_{-1}^{1} \frac{w(\xi) P_n^{(\alpha,\beta)}(\xi)}{x_1 - \xi} \,\mathrm{d}\xi - \mathrm{i}\beta_{\mathrm{G}} P_n^{(\alpha,\beta)}(x_1) w(x_1) = -\frac{(1-\beta_{\mathrm{G}}^2)^{1/2}}{2} P_{n-1}^{(-\alpha,-\beta)}(x_1), \tag{23}$$

$$\int_{-1}^{x_1} P_n^{(\alpha,\beta)}(\xi) w(\xi)\,\mathrm{d}\xi = \frac{-1}{2n} (1-x_1)^{-\beta}(1+x_1)^{-\alpha} P_{n-1}^{(-\beta,-\alpha)}(x_1), \tag{24}$$

for $|x_1| \leqslant 1$, positive integer n and $1 + \alpha + \beta = 0$.

Note that, when $1+\alpha+\beta=0$, the orthogonality of the Jacobi polynomials states that

$$\int_{-1}^{1} P_m^{(-\alpha,-\beta)}(x)P_k^{(-\alpha,-\beta)}(x)(1-x)^{-\alpha}(1+x)^{-\beta}\,\mathrm{d}x = \delta_{km}\,\frac{2\Gamma(k-\alpha+1)\Gamma(k-\beta+1)}{[\Gamma(k+2)]^2}. \tag{25}$$

From equation (22), using equation (25), arrive N complex linear equations for N complex unknowns α_k:

$$\sum_{n=1}^{N}\left[a_n\left(\delta_{n,k}\,\frac{2(1-\beta_{\mathrm{G}}^2)^{1/2}\Gamma(n-\beta)\Gamma(n-\alpha)}{[\Gamma(n+1)]^2}+\frac{(\alpha_1+\alpha_2)}{2n\alpha_0}\,Q_{n,k}\right) + \bar{a}_n\,\frac{\alpha_1^2+2\alpha_3^2-\alpha_1\alpha_2+2\mathrm{i}\alpha_3\alpha_0}{2n\alpha_0\alpha_1}\,H_{n,k}\right] = -\left(\frac{\alpha_3+\mathrm{i}\alpha_0}{\alpha_1}\,t_{1,k}-t_{2,k}\right), \tag{26}$$

where n, $k=1,2,\dots,N$. In the last equation, the complex quantities $Q_{n,k}$, $H_{n,k}$ and $t_{j,k}$ are defined by

$$Q_{n,k} \equiv l\int_{-1}^{1} g(\xi)(1-\xi^2)P_{n-1}^{(-\beta,-\alpha)}(\xi)P_{k-1}^{(-\alpha,-\beta)}(\xi)\,\mathrm{d}\xi, \tag{27 a}$$

$$H_{n,k} \equiv l\int_{-1}^{1} g(\xi)(1-\xi)^{-2\alpha}(1+\xi)^{-2\beta}P_{n-1}^{(-\alpha,-\beta)}(\xi)P_{k-1}^{(-\beta,-\alpha)}(\xi)\,\mathrm{d}\xi, \tag{27 b}$$

$$t_{j,k} \equiv \int_{-1}^{1} T_j(\xi)(1-\xi)^{-\alpha}(1+\xi)^{-\beta}P_{k-1}^{(-\alpha,-\beta)}(\xi)\,\mathrm{d}\xi, \tag{27 c}$$

for n, $k=1,2,\dots,N$ and $j=1,2$.

From equation (18 *b*) and the solutions a_k of the system (26), the dislocation density is expressed as

$$\mathbf{B}(x_1) = \mathbf{EM} = 2\,\mathrm{Re}\left(\begin{bmatrix}\mathrm{i}-\alpha_3/\alpha_0\\ \alpha_1/\alpha_0\end{bmatrix}M_1(x_1)\right) = \sum_{n=1}^{N}2\,\mathrm{Re}\left(\begin{bmatrix}\mathrm{i}-\alpha_3/\alpha_0\\ \alpha_1/\alpha_0\end{bmatrix}a_nP_n^{(\alpha,\beta)}(x_1)w(x_1)\right). \tag{28}$$

From equations (12 *c*) and (28), using equation (24), it follows that the non-dimensional crack-opening displacement is given by

$$\mathbf{u}(x_1) = \sum_{n=1}^{N}\frac{1}{n}\,\mathrm{Re}\left(\begin{bmatrix}\mathrm{i}-\alpha_3/\alpha_0\\ \alpha_1/\alpha_0\end{bmatrix}a_nP_{n-1}^{(-\beta,-\alpha)}(x_1)(1-x_1)^{-\beta}(1+x_1)^{-\alpha}\right). \tag{29}$$

From equation (3), the interface stress outside the crack face, that is, for $|x_1|>1$, is

$$\mathbf{t}(x_1) = \frac{\Lambda_{10}}{\pi}\int_{-1}^{1}\frac{\mathbf{B}(\xi)}{x_1-\xi}\,\mathrm{d}\xi + \mathbf{T}(x_1), \tag{30}$$

where $\mathbf{T}(x_1)$ is the externally applied traction.

3.3. *Stress-intensity factor*

In the literature, there are various definitions of the stress-intensity factor for interface cracks (for example England (1965), Rice and Sih (1965), Willis (1971), Erdogan *et al.* (1973), Hutchinson *et al.* (1987), Rice (1988), Shih and Asaro (1988), Comninou (1990), Suo (1990), Wu (1990), Qu and Li (1991) and Gao *et al.* (1992)). The following discussion will be in line with the work of Erdogan *et al.* (1973) and Suo (1990).

As discussed by Suo (1990) and Ting (1992b), new base vectors $\tilde{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_2$ are defined such that

$$\tilde{\mathbf{e}}_k = \sum_{j=1}^{2}(\Lambda_{10}\mathbf{E})_{jk}\mathbf{e}_j, \qquad k = 1,2, \tag{31}$$

where $\mathbf{e}_j$, $j = 1,2$, are the base vectors for the x_1, x_2 coordinates. Thus, in the $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$ system, the interface stress $\mathbf{t}(x_1)$ for $|x_1| > 1$ is expressed as

$$\mathbf{t} = \sum_{k=1}^{2} t_k\mathbf{e}_k = \sum_{k\,j=1}^{2} t_k(\mathbf{E}^{-1}\Lambda_{10}^{-1})_{jk}\tilde{\mathbf{e}}_j = \sum_{j=1}^{2}\tilde{t}_j\tilde{\mathbf{e}}_j, \tag{32a}$$

where, using equations (28) and (30),

$$\tilde{t}_j = \frac{1}{\pi}\int_{-1}^{1}\frac{M_j(x_1)}{x_1-\xi}\,\mathrm{d}\xi. \tag{32b}$$

Suo (1990) defined a complex stress-intensity factor to be proportional to $\tilde{t}_2$.

Therefore, define the dimensionless stress-intensity factor as

$$\begin{aligned} K_1 + \mathrm{i}K_2 &\equiv \lim_{x\to 1+0}[\tilde{t}_2(x_1-1)^{\beta}(x_1+1)^{-\alpha}] \\ &= \lim_{x\to 1+0}\left[\frac{1}{\pi}\left(\int_{-1}^{1}\frac{M_2(\xi)}{x_1-\xi}\,\mathrm{d}\xi\right)(x_1-1)^{-\beta}(x_1+1)^{-\alpha}\right]. \end{aligned} \tag{33}$$

In terms of the dimensionless components σ_{ij}, the stress-intensity factor given by equation (33) is written as

$$K_1 + \mathrm{i}K_2 = \lim_{x\to 1+0}\left\{\left[\left(\mathrm{i}-\frac{\alpha_3}{\alpha_0}\right)\sigma_{12} + \frac{\alpha_1}{\alpha_0}\sigma_{22}\right](x_1-1)^{-\beta}(x_1+1)^{-\alpha}\right\}. \tag{34}$$

For the isotropic bimaterial, $\alpha_3 = 0$ and $\alpha_1 = \alpha_2 = \alpha_0 = C$. Then,

$$K_1 + \mathrm{i}K_2 = \lim_{x\to 1+0}\{[\sigma_{22} + \mathrm{i}\sigma_{12}](x_1-1)^{-\beta}(x_1+1)^{-\alpha}\} \tag{35a}$$

which, up to a constant factor, coincides with the stress-intensity factor given by Erdogan *et al.* (1973).

For the orthotropic bimaterial, $\alpha_3 = 0$, $\alpha_1 \neq, \alpha_2$, and $\alpha_0 = (\alpha_1\alpha_2)^{1/2}$ leading to

$$K_1 + \mathrm{i}K_2 = \lim_{x\to 1+0}\left\{\left[\mathrm{i}\sigma_{12} + \left(\frac{\alpha_1}{\alpha_2}\right)^{1/2}\sigma_{22}\right](x_1-1)^{-\beta}(x_1+1)^{-\alpha}\right\} \tag{35b}$$

which, apart from a constant factor, is the same as the stress-intensity factor given by Suo (1990).

Solutions for the dislocation density and the stresses on the x_1 axis are oscillatory at the crack tips, $x_1 = \pm 1$. On the other hand, the above-defined stress-intensity factor is non-oscillatory. It is calculated from equation (33) and gives

$$K_1 + \mathrm{i}K_2 = (1-\beta_{\mathrm{G}}^2)^{1/2} \sum_{n=1}^{N} \bar{a}_n P_n^{(\beta,\alpha)}(1), \tag{36}$$

where the following identity (Erdelyi *et al.* 1953) is required:

$$\begin{aligned}\int_{-1}^{1} \frac{(1-t)^{\alpha}(1+t)^{\beta} P_n^{(\alpha,\beta)}(t)}{(x-t)}\,\mathrm{d}t = \frac{-\pi}{2\sin(\pi\alpha)} & P_n^{(\alpha,\beta)}(x)(x-1)^{\alpha}(x+1)^{\beta} \\ & + 2^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} \\ & \times F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{1-x}{2}\right),\end{aligned} \tag{37}$$

where $x = 1 + \epsilon$ with suitable $\epsilon > 0$, F is the hypergeometric function, and the Jacobi polynomial for $x > 1$ is understood to be

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!}(x-1)^{-\alpha}(x+1)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(x-1)^{n+\alpha}(x+1)^{n+\beta}]. \tag{38}$$

3.4. *Energy release rate*

It is emphasized that, for the interface cracks, various field quantities involve oscillatory singularities. On the other hand, the energy release rates are always non-oscillatory (Nemat-Nasser and Ni 1993). Hence, the energy release rate is a better physical quantity to describe the state of the interface cracks. For the bridged interface crack, the energy release rate is given by

$$G = \frac{\pi}{16}|K_1 + \mathrm{i}K_2|^2. \tag{39}$$

In terms of the series solution, the energy release rate becomes

$$G = \frac{\pi}{16}(1-\beta_{\mathrm{G}}^2)\left|\sum_{n=1}^{N} a_n P_n^{(\alpha,\beta)}(1)\right|^2. \tag{40}$$

3.5. *Example*

As an example, consider a bridged interface crack in dissimilar isotropic materials. The non-dimensional matrices $\boldsymbol{\Lambda}_{10}$ and $\boldsymbol{\Lambda}_{20}$ have now become

$$\boldsymbol{\Lambda}_{10} = \mathbf{I} \tag{41 a}$$

and

$$\boldsymbol{\Lambda}_{20} = \begin{bmatrix} 0 & \beta_{\mathrm{D}} \\ -\beta_{\mathrm{D}} & 0 \end{bmatrix}, \tag{41 b}$$

where $\mathbf{I}$ is the unit matrix and β_{D} is the Dundurs constant defined in equation (6 *c*).

The basic equation (14) becomes

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\mathbf{B}(\xi)}{x_1-\xi}\,\mathrm{d}\xi+\begin{bmatrix}0 & \beta_{\mathrm{D}}\\ -\beta_{\mathrm{D}} & 0\end{bmatrix}\mathbf{B}(x_1)+lg(x_1)\int_{-1}^{x_1}\mathbf{B}(\xi)\,\mathrm{d}\xi+\mathbf{T}(x_1)=\mathbf{0}. \tag{42}$$

Contrary to the general anisotropic case, in this special case, the system (17) is decoupled by diagonalizing the matrix Λ_{20} leading to

$$\frac{1}{\pi}\int_{-1}^{1}\frac{M_1(\xi)}{x_1-\xi}\,\mathrm{d}\xi-\mathrm{i}\beta_{\mathrm{D}}M_1(x_1)+lg(x_1)\int_{-1}^{x_1}M_1(\xi)\,\mathrm{d}\xi=-t_1(\xi), \tag{43 a}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{M_2(\xi)}{x_1-\xi}\,\mathrm{d}\xi+\mathrm{i}\beta_{\mathrm{D}}M_2(x_1)+lg(x_1)\int_{-1}^{x_1}M_2(\xi)\,\mathrm{d}\xi=-\bar{t}_1(\xi), \tag{43 b}$$

with auxiliary conditions

$$\int_{-1}^{1}M_1(\xi)\,\mathrm{d}\xi=\int_{-1}^{1}M_2(\xi)\,\mathrm{d}\xi=0, \tag{43 c}$$

where

$$\mathbf{M}(x_1)=\tfrac{1}{2}\begin{bmatrix}-\mathrm{i} & 1\\ \mathrm{i} & 1\end{bmatrix}\mathbf{B}(x_1) \tag{43 d}$$

and

$$\mathbf{t}=\begin{pmatrix}t_1(\xi)\\ \bar{t}_1(\xi)\end{pmatrix}=\tfrac{1}{2}\begin{bmatrix}-\mathrm{i} & 1\\ \mathrm{i} & 1\end{bmatrix}\mathbf{T}(x_1). \tag{43 e}$$

Consider the case where the external traction $\mathbf{T}$ is uniform. them, since the decoupled linear equations (43 *a*) and (43 *b*) are each other's complex conjugate, it suffices to consider the following equation:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{F(\xi)}{x_1-\xi}\,\mathrm{d}\xi-\mathrm{i}\beta_{\mathrm{D}}F(x_1)+lg(x_1)\int_{-1}^{x_1}F(\xi)\,\mathrm{d}\xi=-1, \tag{44 a}$$

together with the auxiliary condition

$$\int_{-1}^{1}F(\xi)\,\mathrm{d}\xi=0. \tag{44 b}$$

If $F(x_1)$ is the solution of equations (44 *a*) and (44 *b*), then

$$M_1(x_1)=t_1F(x_1) \tag{45 a}$$

and

$$M_2(x_1)=\overline{M_1(x_1)}=\overline{t_1F(x_1)} \tag{45 b}$$

are the solutions of equations (43 *a*) and (43 *b*) respectively. From these, the dislocation density vector is recovered as

$$\mathbf{B}(x_1)=\begin{bmatrix}T_1 & -T_2\\ T_2 & T_1\end{bmatrix}\begin{pmatrix}\mathrm{Re}\,[F(x_1)]\\ \mathrm{Im}\,[F(x_1)]\end{pmatrix}. \tag{46}$$

Employ the method described in § 3.2, approximate $F(x_1)$ by

$$F(x_1)\approx\sum_{n=1}^{N}d_nP_n^{(\alpha,\beta)}(x_1)w(x_1) \tag{47}$$

and determine the complex coefficients d_n, $n = 1, 2, \dots, N$, by solving the following equations;

$$\sum_{n=1}^{N} d_n \left[(1-\beta_D^2)^{1/2} P_{n-1}^{(-\alpha,-\beta)}(x_1) + \frac{lg(x_1)}{n}(1-x_1)^{-\beta}(1+x_1)^{-\alpha} P_{n-1}^{(-\beta,-\alpha)}(x_1) \right] = 2. \quad (48)$$

Using the orthogonality of the Jacobi polynomials, equation (48) is transformed to a system of linear equations

$$\sum_{n=1}^{N} \left[d_n \left(\delta_{nk} \frac{2(1-\beta_D^2)^{1/2} \Gamma(n-\beta)\Gamma(n-\alpha)}{[\Gamma(n+1)]^2} + \frac{1}{n} Q_{nk} \right) \right] = 4\Gamma(1-\alpha)\Gamma(1-\beta)\delta_{k1}, \quad (49\,a)$$

where $n, k = 1, 2, \dots, N$, δ_{nm} is the Kronecker delta, and the complex quantity $Q_{n,k}$ is defined by

$$Q_{n,k} \equiv l \int_{-1}^{1} g(\xi)(1-\xi^2) P_{n-1}^{(-\beta,-\alpha)}(\xi) P_{k-1}^{(-\alpha,-\beta)}(\xi)\, d\xi. \quad (49\,b)$$

The stress-intensity factor is then obtained as

$$K_1 + iK_2 = (1-\beta_D^2)^{1/2}(T_2 + iT_1) \sum_{n=1}^{N} \bar{d}_n P_n^{(\beta,\alpha)}(1). \quad (50)$$

The energy release rate is given by

$$G = \frac{\pi(1-\beta_D^2)}{16}(T_2^2 + T_1^2) \left| \sum_{n=1}^{N} d_n P_n^{(\alpha,\beta)}(1) \right|^2. \quad (51)$$

For illustration, we present numerical results for the cases of fully and partially bridged interface cracks. Here, we use the Dundurs parameter $\beta_D = 0.4854$, as used by Comninou (1977), and the external tractions $T_2 = 1$ and $T_1 = 0$. Assume that the non-dimensional crack length take the values $l = 0.1$, 1 and 10 respectively.

For fully bridged cracks and uniform bridging force with $g(x_1) = 1$, the calculated stress-intensity factors are listed in table 1.

The non-dimensional parameter $l = K_0 L / C$ may be used to measure the effectiveness of a given crack bridging; here C is the Dundurs constant defined in equation (6 *a*). When $l = 0$, there is no bridging and the non-dimensional stress-intensity factor is independent of the crack length L. Small values of l correspond to weak bridging effects, that is small K_0 or great material mismatch (large C), when L is fixed. On the other hand, for fixed values of K_0 and C, the parameter l increases with increasing crack length L, leading to decreasing non-dimensional stress-intensity factor.

Table 1. Fully bridged interface cracks.

l	K_1	K_2	G
0.1	0.924	−0.336	0.190
1.0	0.578	−0.311	0.085
10	0.165	−0.186	0.012

Figures 1 and 2 show the dependence of the non-dimensional mode I stress-intensity factor K_1 and mode II stress-intensity factor K_2 respectively on the non-dimensional crack length l, when $g(x_1) = 1$, 10 and 100. Figure 3 gives the dependence of the dimensionless energy release rate G on the non-dimensional crack length l, when $g(x_1) = 1$, 10 and 100.

For the partially bridged crack cases, with the cracks uniformly bridged from $-l+0.1$ to $l-0.1$, over the bridged interval, $g(x_1) = 1$, the stress-intensity factors are then given in table 2.

Comparing table 2 with table 1, it is seen that for the same non-dimensional parameter l, the stress-intensity factors and the energy release rates for the partially bridged crack are greater than the corresponding values for the fully bridged crack.

For non-uniformly bridging forces, assume that the bridging force is defined by a quadratic form as used by Nemat-Nasser and Hori (1987):

$$g(x_1) = \begin{cases} \dfrac{4}{(l-c_1)^2}(l-|x_1|)(|x_1|-c_1) & \text{for } c_1 < |x_1| < l, \\ 0 & \text{otherwise.} \end{cases} \tag{52}$$

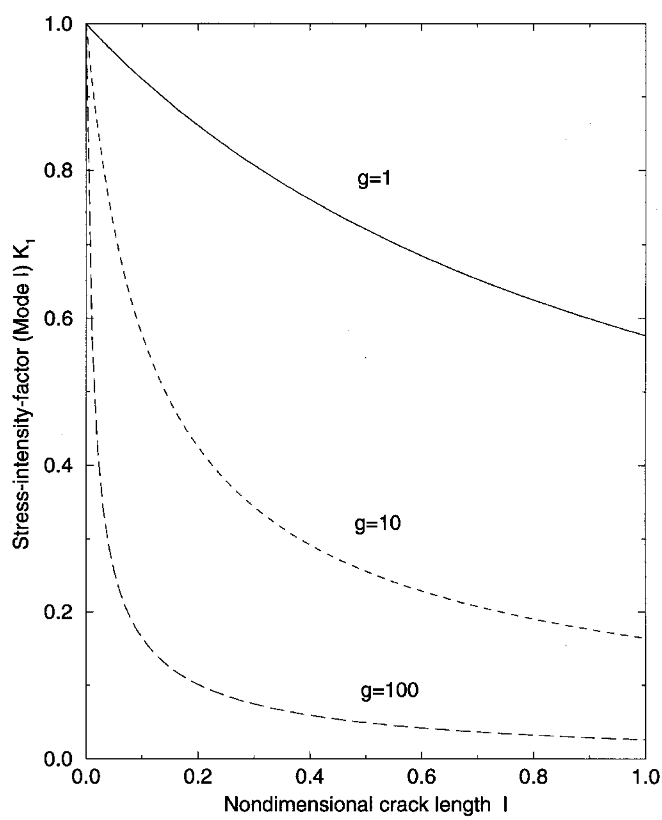

Figure 1. Dependence of the mode I non-dimensional stress-intensity factor K_1 on the non-dimensional crack length l for the bridged interface crack in an isotropic bimaterial with the Dundurs constant $\beta_D = 0.4854$, when the external loadings are $T_1 = 0$ and $T_2 = 1$ and the bridging forces are $g(x_1) \equiv 1$, 10 and 100.

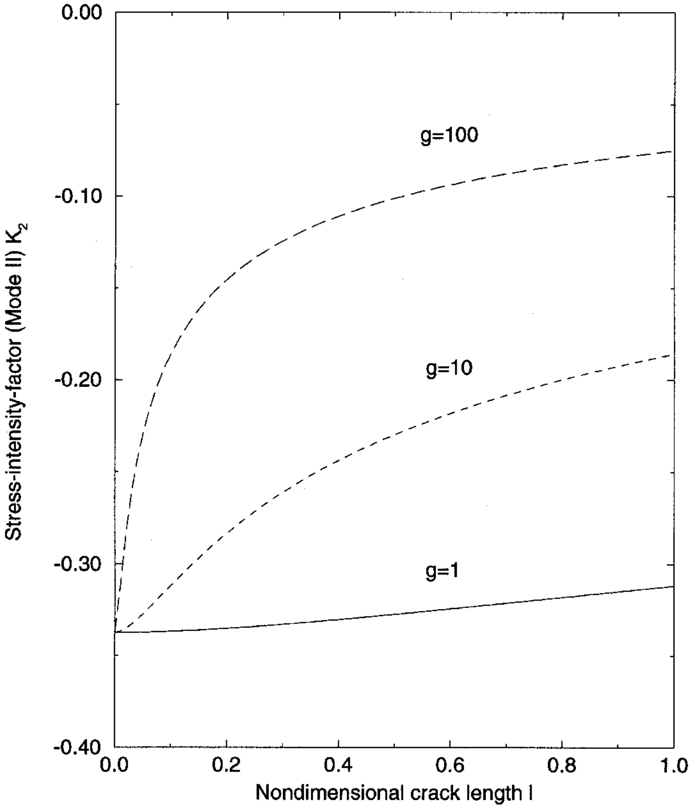

Figure 2. Dependence of the mode II non-dimensional stress-intensity factor K_2 on the non-dimensional crack length l for the bridged interface crack in an isotropic bimaterial with the Dundurs constant $\beta_D = 0.4854$, when the external loadings are $T_1 = 0$ and $T_2 = 1$ and the bridging forces are $g(x_1) \equiv 1$, 10 and 100.

Then, the crack is partially bridged over $c_1 < |x_1| < l$, where l is the non-dimensional crack length and c_1 is a constant, specifying the bridging interval; is the smaller the c_1, the larger is the bridging interval. For different crack lengths l and different values of c, the calculated stress-intensity factors are listed in table 3.

Figure 4 shows the comparison between the dimensionless energy release rate of the bridged interface crack in a bimaterial, that is $\beta_D = 0.4854$, and the bridged crack in a homogeneous material, that is $\beta_D = 0$. The numerical results suggest that, for a bridged interface crack, the bridging forces enhance the material toughness effectively. When there is no fibre bridging, the energy release rates for the crack in the homogeneous material and the interface crack in the bimaterial take the values 0.196 350 and 0.218 705 respectively. This implies that the interface crack is more apt to open. However, when the bridging forces increase, that is either K_0 or $g(x_1)$ becomes greater, the differences between the non-dimensional energy release rates of the crack and the interface crack become insignificant, that is the material mismatch effect in the case of the interface crack is markedly overcome by the bridging forces. While for the fully bridged crack, it is possible to obtain the bridging strength above which the material mismatch effect is insignificant, for a partially bridged

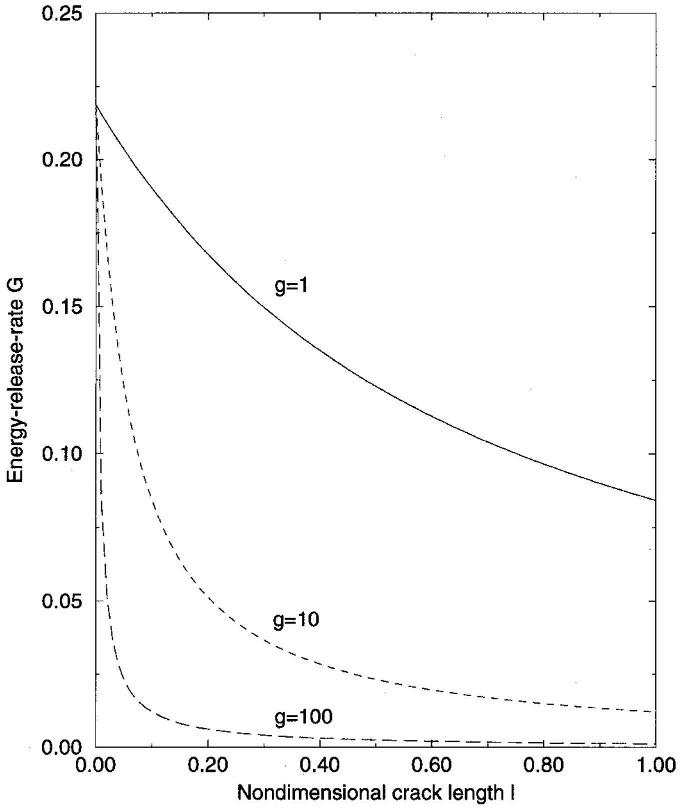

Figure 3. Dependence of the non-dimensional energy release rate G on the non-dimensional crack length l for the bridged interface crack in an isotropic bimaterial with the Dundurs constant $\beta_D = 0.4854$, when the external loadings are $T_1 = 0$ and $T_2 = 1$ and the bridging forces are $g(x_1) \equiv 1$, 10 and 100.

Table 2. Partially bridged interface cracks.

l	K_1	K_2	G
1.0	0.621	−0.318	0.096
10	0.273	−0.233	0.025

Table 3. Cracks with non-uniformly bridging forces.

l	c_1	K_1	K_2	G
1.0	0.25	0.724	−0.329	0.124
1.0	0.50	0.792	−0.333	0.145
1.0	0.75	0.881	−0.336	0.175
10	0.25	0.299	−0.249	0.030
10	0.50	0.344	−0.270	0.038
10	0.75	0.466	−0.303	0.061

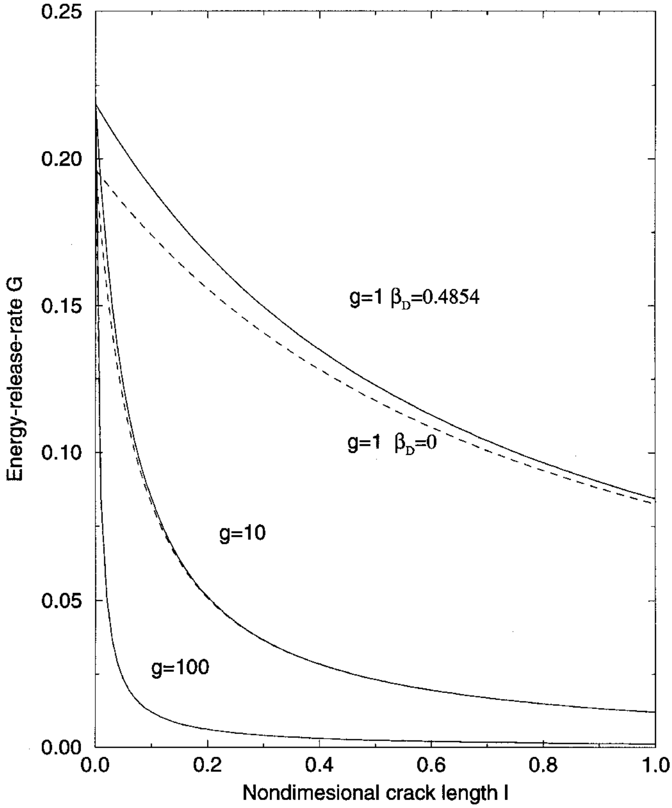

Figure 4. Comparison of the non-dimensional energy release rates G for the bridged interface crack in an isotropic bimaterial with the Dundurs constant $\beta_D = 0.4854$ (——) and the bridged crack in a homogeneous isotropic material with $\beta_D = 0$ (— —), when the external loadings are $T_1 = 0$ and $T_2 = 1$ and the bridging forces are $g(x_1) \equiv 1$, 10 and 100.

crack, this will also depend on the bridging length relative to the crack length. For given bridging strength and crack length, there may or may not exist a bridging length above which the effect of the material mismatch becomes insignificant.

§4. Conclusion

Bridged interface cracks in anisotropic bimaterials are considered. It is assumed that the bridging force is an explicit functional of the crack-opening displacement. Based on the equilibrium, the governing equations for the dislocation density which defines the crack-opening displacement is established. The involved elastic parameters for some frequently used bimaterials are listed. Special attention is then given to the cases where the bridging forces depend linearly on the crack-opening displacement. A system of Cauchy singular integral equations for the dislocation density is derived. Series solutions in terms of the Jacobi polynomials are obtained for the dislocation density, the crack opening displacement and the stresses. The non-oscillatory solutions for the stress-intensity factor and the energy release rate are discussed.

For illustration, an example of a bridged interface crack in an isotropic bimaterial is examined in detail. Numerical results are obtained to illustrate the strengthening effect of bridging fibres. With the increasing bridging forces, the energy release rate for the bridged interface crack in a bimaterial rapidly tends to that of the bridged crack in a homogeneous material.

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