Interaction of a harmonic wave with a dynamically transforming inhomogeneity

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The elastodynamic response of the transformation-toughened ceramics under a time-harmonic stress wave is investigated. A phenomenological model is proposed to describe the situation, which involves the interaction between an incident stress wave and a dynamic inhomogeneity with a stress-induced martensitic transformation. The most important assumption made in this model is that the stress-induced transformation can be treated as completely reversible. The solution for this model is obtained by combining solutions to a scattering problem, a dynamic inhomogeneity problem, and a static inhomogeneity problem. An exact closed form solution is obtained for the dynamic inhomogeneity problem. The numerical results for the zirconia-toughened ceramics suggest that, under the high-frequency dynamic loading, the transformation-toughened ceramics might lose its toughness due to a relatively large tension field caused by the dynamically transforming zirconia particle.

I. INTRODUCTION

Zirconia toughened ceramics is a remarkable material, which has a high strength, a high elastic modulus, a good thermal shock resistance, and an improved toughness, etc.[1–3] Most of the good qualities are common in many ceramic materials. The improved toughness is, however, the distinguished character of the zirconia-toughened ceramics, which is made possible through the transformation-toughening by zirconia particles. The transformation-toughening utilizes the stress-induced phase transformation of zirconia particles (tetragonal→monoclinic), which is accompanied by a volumetric expansion. There are mainly three mechanisms which contribute to the transformation-toughening:[1,2,4] (i) the energy absorption by the stress-induced martensitic transformation of zirconia particles (t-ZrO_2) near the crack tip, (ii) the nucleation of matrix cracks around the transformed particles (m-ZrO_2), and (iii) the deflection of cracks near zirconia particles (m or t).

A considerable amount of research has been done to understand the mechanisms of the transformation-toughening by zirconia particles.[1,2,4–12] All of these studies have been for the transformation-toughening of ceramics under either a static or a quai-static environment. To the best of the authors' knowledge, however, no such work has been performed for the transformation-toughening of ceramics under a fully dynamic environment. By the fully dynamic environment is meant one of the following situations: (i) the dynamic crack propagation in a transformation-toughened ceramics under a constant loading, (ii) the quasi-static crack propagation in a transformation-toughened ceramics under a dynamic loading, (iii) the dynamic crack propagation in a transformation-toughened ceramics under a dynamic loading. None of these problems is simple. In this paper, as a prerequisite for solving problems (ii) and (iii), the dynamic behavior of the transformation-toughened ceramics under a dynamic loading is investigated.

No cracks are involved in our problem. The dynamic response of the zirconia-toughened ceramics under a dynamic loading is studied. A couple of assumptions and simplifications are made to model the problem. The most important assumption is that the stress-induced martensitic transformation of the zirconia particle can be treated as completely reversible.[11,13] Also by assuming the dilute concentration of zirconia particles, we neglect the multiparticle interaction of zirconia particles with the incident time-harmonic stress wave. Furthermore, we only consider the axisymmetric dynamic transformation strain. Since we are primarily concerned with a local elastic field in and around the zirconia particle, with the above assumptions, the zirconia-toughened ceramics is modeled as an infinite elastic medium containing a single spherical dynamic inhomogeneity under the time-harmonic stress wave. In our recent paper,[14] we have treated a spherical dynamic inclusion problem in an attempt to understand the fundamental feature of the dynamic behavior of the transformation-toughened ceramics. In that paper, the interaction between the zirconia particle and the incident stress wave was neglected. Also the zirconia particle was treated as an inclusion (the same material as the matrix) instead of an inhomogeneity, where the elastic property is different from that of the matrix. Thus our present study stands as an extension of our previous study[14] as long as the dynamic transformation strain is axisymmetric.

The solution for the phenomenological model is obtained by combining solutions to a scattering problem, a dynamic inhomogeneity problem, and a static inhomogeneity problem. Scattering by a spherical particle has been treated by Pao and Mow.[15] An exact closed form solution

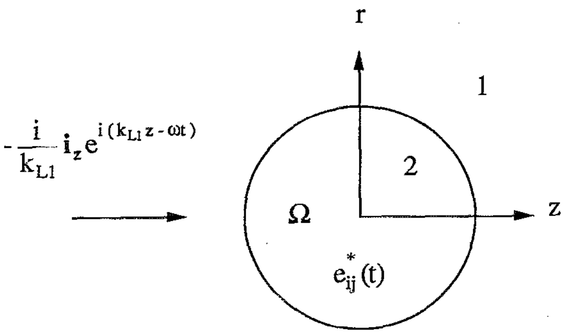


FIG. 1. Plane longitudinal wave incident on a dynamically transforming spherical inhomogeneity.

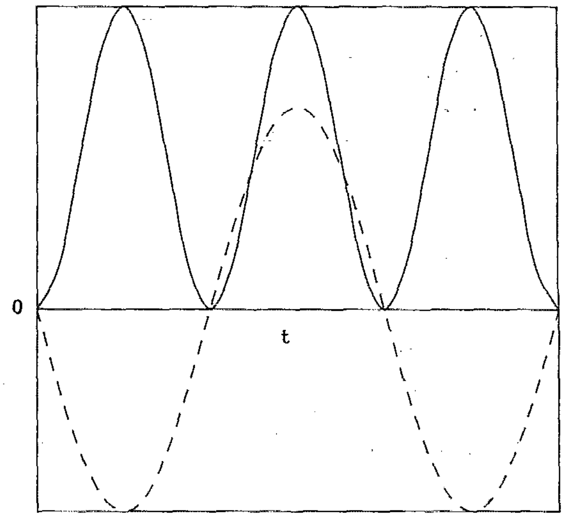


FIG. 2. Phase transformation (solid line) induced by an incident stress wave (dashed line).

is obtained for the dynamic inhomogeneity problem by following their approach. Numerical results are shown for the zirconia-toughened alumina ceramics. These numerical results suggest that, under high-frequency dynamic loading, the transformation-toughened ceramics might lose some of their toughness due to a relatively large tension field caused by the dynamically transforming zirconia particle.

II. STATEMENT OF THE PROBLEM

The geometry of the problem is shown in Fig. 1. The matrix and the inhomogeneity are denoted as 1 and 2, respectively. A longitudinal wave is incident upon the spherical inhomogeneity Ω. A phase transformation of the inhomogeneity Ω is induced by the incident longitudinal stress wave. We model this phase transformation by the following eigenstrain $e^*_{ij}(t)$.

$$e^*_{ij}(t)=\frac{1}{2}e^*_{ij}[e^{-2i\omega t}+1], \tag{1}$$

where ω is the angular frequency of the incident wave. The reason for the doubled frequency 2ω for the dynamic inhomogeneity is that the phase transformation is modeled to occur in both tension and compression stress fields caused by the incident stress wave (see Fig. 2). Furthermore, it is assumed here that the tension and compression fields produce the same amount of transformation strain, which may not be the case in general. It is also implicitly assumed in Eq. (1) that the transformation strain is proportional to the square of the value of the incident longitudinal stress. It should be also mentioned here that only the real part of $e^*_{ij}(t)$ is taken as physically meaningful. The assumed reversibility of our model (1) for the eigenstrain $e^*_{ij}(t)$ is partially supported by the experimental observations by Marshall and James[11] and by Reyes-Morel *et al.*[13] The governing equations are

$$\sigma^1_{ij,j}=\rho_1\ddot{u}^1_i, \tag{2a}$$

$$\sigma^1_{ij}=C^1_{ijkl}e^1_{kl} \quad \text{in } \mathbf{R}^3-\Omega, \tag{2b}$$

$$e^1_{kl}=\frac{1}{2}(u^1_{k,l}+u^1_{l,k}), \tag{2c}$$

$$\sigma^2_{ij,j}=\rho_2\ddot{u}^2_i, \tag{3a}$$

$$\sigma^2_{ij}=C^2_{ijkl}(e^2_{kl}-e^*_{kl}) \quad \text{in } \Omega, \tag{3b}$$

$$e^2_{kl}=\frac{1}{2}(u^2_{k,l}+u^2_{l,k}), \tag{3c}$$

where ρ_m and C^m_{ijkl} are a mass density and an elasticity tensor of the material m (= 1 or 2), respectively. In all of the following calculations, the material isotropy is assumed for both the material 1 and 2. The boundary conditions are

$$\left.\begin{array}{l} u^1_i=u^2_i \\ \sigma^1_{ij}n_j=\sigma^2_{ij}n_j \end{array}\right. \quad \text{at } r=a, \tag{4}$$

where n_j is a unit normal to the surface of the spherical inhomogeneity Ω, and a is the radius of the sphere. The incident longitudinal wave is given by

$$\mathbf{u}^{in}=-\frac{i}{k_{L1}}\mathbf{i}_z e^{i(k_{L1}z-\omega t)}, \tag{5}$$

where $\mathbf{i}_z$ is a unit vector in the z direction, and

$$k_{L1}=\frac{\omega}{c_{L1}}, \quad c_{L1}=\sqrt{2\mu_1+\lambda_1/\rho_1}. \tag{6}$$

The constant factor in (5) is chosen so as to make the strain amplitude derived from (5) unity.

Due to the linearity of the governing equations (2) and (3), our problem can be decomposed into three subproblems: scattering, dynamic inhomogeneity, and static inhomogeneity. In the following sections these subproblems will be treated in detail.

III. SCATTERING

Scattering of a plane longitudinal wave by a spherical inhomogeneity has been treated by Pao and Mow.[15] In this section, we summarize their results.

Since the incident longitudinal wave has the time factor $e^{-i\omega t}$, the elastic field due to scattering has also the same time factor. In the following analysis, we omit this time factor, and concentrate on the spatial part of the solution. Also, since our scattering problem has an axisymmetry with respect to the z axis, the displacement field in spherical coordinates (ρ,θ,ϕ), can be expressed in terms of two displacement potentials Φ and Ψ as

$$\mathbf{u}^m=\nabla\Phi^m+\nabla\times\left(e_\phi\frac{\partial\Psi^m}{\partial\theta}\right),\quad m=1 \text{ or } 2, \tag{7}$$

where e_ϕ is a base vector for the ϕ axis, and the potentials satisfy the following equations:

$$\nabla^2\Phi^m+k_{Lm}^2\Phi^m=0,\quad \nabla^2\Psi^m+k_{Tm}^2\Psi^m=0, \tag{8}$$

where

$$k_{Lm}=\omega/c_{Lm},\quad c_{Lm}=\sqrt{2\mu_m+\lambda_m/\rho_m},$$

$$k_{Tm}=\omega/c_{Tm},\quad c_{Tm}=\sqrt{\mu_m/\rho_m}. \tag{9}$$

In material 1, the potentials are decomposed into the incident and reflected waves. Thus the expansions of the potentials in terms of spherical harmonics are given by

$$\Phi^1=\Phi^i+\Phi^r,\quad \Psi^1=\Psi^r,$$

$$\Phi^i=\Phi_0\sum_{n=0}^{\infty}(2n+1)i^n j_n(k_{L1}r)P_n(\cos\theta),$$

$$\Phi_0=-\frac{1}{k_{L1}^2},$$

$$\Phi^r=\sum_{n=0}^{\infty}A_n h_n(k_{L1}r)P_n(\cos\theta),$$

$$\Psi^r=\sum_{n=0}^{\infty}B_n h_n(k_{T1}r)P_n(\cos\theta),$$

$$\Phi^2=-\sum_{n=0}^{\infty}C_n j_n(k_{L2}r)P_n(\cos\theta),$$

$$\Psi^2=-\sum_{n=0}^{\infty}D_n j_n(k_{T2}r)P_n(\cos\theta), \tag{10}$$

where j_n is the spherical Bessel function of the first kind, h_n is the spherical Hankel function of the first kind, and P_n is the Legendre polynomials. Here A_n, B_n, C_n and D_n are the expansion coefficients, which are to be determined by the boundary conditions (4). From (7) and (10), the displacement field and the stress field in each material are given by

$$u_{r1}=\frac{1}{r}\sum_{n=0}^{\infty}(-\Phi_0\epsilon_1+A_n\epsilon_{11}+B_n\epsilon_{12})P_n(\cos\theta),$$

$$u_{\theta1}=\frac{1}{r}\sum_{n=0}^{\infty}(-\Phi_0\epsilon_2+A_n\epsilon_{21}+B_n\epsilon_{22})\frac{dP_n}{d\theta}(\cos\theta),$$

$$\sigma_{rr1}=\frac{2\mu_1}{r^2}\sum_{n=0}^{\infty}(-\Phi_0\epsilon_3+A_n\epsilon_{31}+B_n\epsilon_{32})P_n(\cos\theta),$$

$$\sigma_{r\theta1}=\frac{2\mu_1}{r^2}\sum_{n=0}^{\infty}(-\Phi_0\epsilon_4+A_n\epsilon_{41}+B_n\epsilon_{42})\frac{dP_n}{d\theta}(\cos\theta) \tag{11}$$

$$u_{r2}=-\frac{1}{r}\sum_{n=0}^{\infty}(C_n\epsilon_{13}+D_n\epsilon_{14})P_n(\cos\theta),$$

$$u_{\theta2}=-\frac{1}{r}\sum_{n=0}^{\infty}(C_n\epsilon_{23}+D_n\epsilon_{24})\frac{dP_n}{d\theta}(\cos\theta),$$

$$\sigma_{rr2}=-\frac{2\mu_2}{r^2}\sum_{n=0}^{\infty}(C_n\epsilon_{33}+D_n\epsilon_{34})P_n(\cos\theta),$$

$$\sigma_{r\theta2}=-\frac{2\mu_2}{r^2}\sum_{n=0}^{\infty}(C_n\epsilon_{43}+D_n\epsilon_{44})\frac{dP_n}{d\theta}(\cos\theta), \tag{12}$$

where

$$\epsilon_1=-i^n(2n+1)[nj_n(k_{L1}r)-k_{L1}rj_{n+1}(k_{L1}r)],$$

$$\epsilon_2=-i^n(2n+1)j_n(k_{L1}r),$$

$$\epsilon_3=-i^n(2n+1)[(n^2-n-\tfrac{1}{2}(k_{T1}r)^2)j_n(k_{L1}r) + 2k_{L1}rj_{n+1}(k_{L1}r)],$$

$$\epsilon_4=-i^n(2n+1)[(n-1)j_n(k_{L1}r) - k_{L1}rj_{n+1}(k_{L1}r)],$$

$$\epsilon_{11}=nh_n(k_{L1}r)-k_{L1}rh_{n+1}(k_{L1}r),$$

$$\epsilon_{21}=h_n(k_{L1}r),$$

$$\epsilon_{31}=[n^2-n-\tfrac{1}{2}(k_{T1}r)^2]h_n(k_{L1}r) + 2k_{L1}rh_{n+1}(k_{L1}r),$$

$$\epsilon_{41}=(n-1)h_n(k_{L1}r)-k_{L1}rh_{n+1}(k_{L1}r),$$

$$\epsilon_{12}=-n(n+1)h_n(k_{T1}r),$$

$$\epsilon_{22}=-(n+1)h_n(k_{T1}r)+k_{T1}rh_{n+1}(k_{T1}r), \tag{13}$$

$$\epsilon_{32}=-n(n+1)[(n-1)h_n(k_{T1}r) - k_{T1}rh_{n+1}(k_{T1}r)],$$

$$\epsilon_{42}=-\left(n^2-1-\frac{1}{2}(k_{T1}r)^2\right)h_n(k_{T1}r) - k_{T1}rh_{n+1}(k_{T1}r),$$

$$\epsilon_{13}=nj_n(k_{L2}r)-k_{L2}rj_{n+1}(k_{L2}r),$$

$$\epsilon_{23}=j_n(k_{L2}r),$$

$$\epsilon_{33}=[n^2-n-\tfrac{1}{2}(k_{T2}r)^2]j_n(k_{L2}r)+2k_{L2}rj_{n+1}(k_{L2}r),$$

$$\epsilon_{43}=(n-1)j_n(k_{L2}r)-k_{L2}rj_{n+1}(k_{L2}r),$$

$$\epsilon_{14}=-n(n+1)j_n(k_{T2}r),$$

$$\epsilon_{24}=-(n+1)j_n(k_{T2}r)+k_{T2}r_{n+1}(k_{T2}r),$$

$$\epsilon_{34}=-n(n+1)[(n-1)j_n(k_{T2}r)$$

$$- k_{T2} r j_{n+1}(k_{T2} r)],$$

$$\epsilon_{44} = -[n^2 - 1 - \tfrac{1}{2}(k_{T2} r)^2] j_n(k_{T2} r) - k_{T2} r j_{n+1}(k_{T2} r).$$

Substituting (11) and (12) into (4) we obtain

$$\begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & pE_{33} & pE_{34} \\ E_{41} & E_{42} & pE_{43} & pE_{44} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{bmatrix} = \Phi_0 \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix}, \tag{14}$$

where

$$E_{ij} = (\epsilon_{ij})_{r=a}, \quad E_i = (\epsilon_i)_{r=a}, \tag{15}$$

and

$$p = \mu_2/\mu_1. \tag{16}$$

IV. DYNAMIC INHOMOGENEITY

In this section, we will show that the exact closed form solution can be obtained for the inhomogeneity problem with the dynamic part of the eigenstrain $e^*_{ij}(t)$ given by (1), if the following condition is met:

$$e^*_{11} = e^*_{22}, \quad e^*_{ij} = 0 \quad (i \neq j). \tag{17}$$

Equation (17) ensures that the dynamic inhomogeneity problem is axisymmetric with respect to the z axis. Thus this problem can be solved in exactly the same manner as the scattering problem in the previous section except a few changes.

If the eigenstrain is no longer axisymmetric, then the vector representation for the displacement (7) requires four displacement potentials [i.e., Φ and (Ψ_1,Ψ_2,Ψ_3)]. Thus Eq. (7) has to be modified. Furthermore, the full expansions of the potentials Φ and Ψ_i $(i=1,2,3)$ with respect to (r,θ,ϕ) coordinates are required instead of the expansions with respect to (r,θ) coordinates only [see Equation (10)]. Although the procedure for the general case is still straightforward, the algebra becomes substantially more complicated. In this paper, however, only the axisymmetric case will be considered.

The elastic field due to the dynamic part of the eigenstrain (1) has the same time factor $e^{-2i\omega t}$. In the following, we omit this time factor, and concentrate on the spatial part of the solution. By noting that now, instead of the incident wave, the dynamic eigenstrain is present, the displacement field and the stress field in each material can be expressed as:

$$u_{r1} = \frac{1}{r} \sum_{n=0}^{\infty} (A_n \epsilon_{11} + B_n \epsilon_{12}) P_n(\cos\theta),$$

$$u_{\theta 1} = \frac{1}{r} \sum_{n=0}^{\infty} (A_n \epsilon_{21} + B_n \epsilon_{22}) \frac{dP_n}{d\theta}(\cos\theta),$$

$$\sigma_{rr1} = \frac{2\mu_1}{r^2} \sum_{n=0}^{\infty} (A_n \epsilon_{31} + B_n \epsilon_{32}) P_n(\cos\theta),$$

$$\sigma_{r\theta 1} = \frac{2\mu_1}{r^2} \sum_{n=0}^{\infty} (A_n \epsilon_{41} + B_n \epsilon_{42}) \frac{dP_n}{d\theta}(\cos\theta), \tag{18}$$

$$u_{r2} = -\frac{1}{r} \sum_{n=0}^{\infty} (C_n \epsilon_{13} + D_n \epsilon_{14}) P_n(\cos\theta),$$

$$u_{\theta 2} = -\frac{1}{r} \sum_{n=0}^{\infty} (C_n \epsilon_{23} + D_n \epsilon_{24}) \frac{dP_n}{d\theta}(\cos\theta),$$

$$\sigma_{rr2} = -\frac{2\mu_2}{r^2} \sum_{n=0}^{\infty} (C_n \epsilon_{33} + D_n \epsilon_{34}) P_n(\cos\theta) - \sigma^*_{rr2},$$

$$\sigma_{r\theta 2} = -\frac{2\mu_2}{r^2} \sum_{n=0}^{\infty} (C_n \epsilon_{43} + D_n \epsilon_{44}) \frac{dP_n}{d\theta}(\cos\theta) - \sigma^*_{r\theta 2}, \tag{19}$$

where ϵ_{ij} $(1 \leqslant i,j \leqslant 4)$ are exactly same as those in (13), except that now the wave numbers k_{Lm} and k_{Tm} are given by

$$k_{Lm} = \frac{2\omega}{c_{Lm}}, \quad k_{Tm} = \frac{2\omega}{c_{Tm}}, \tag{20}$$

and A_n, B_n, C_n, and D_n are the expansion coefficients which are to be determined by the boundary conditions (4). σ^*_{rr2} and $\sigma^*_{r\theta 2}$ are the eigenstresses in the spherical coordinates derived from (17), and given by

$$\sigma^*_{rr2} = -\tfrac{2}{3}(\sigma^*_{11} - \sigma^*_{33}) P_2(\cos\theta) + \tfrac{1}{3}(2\sigma^*_{11} + \sigma^*_{33}) P_0(\cos\theta),$$

$$\sigma^*_{r\theta 2} = -\frac{1}{3}(\sigma^*_{11} - \sigma^*_{33}) \frac{dP_2}{d\theta}(\cos\theta), \tag{21}$$

where

$$\sigma^*_{ij} = 2\mu_2 a^*_{ij} + \lambda_2 a^*_{kk} \delta_{ij}, \quad a^*_{ij} = \tfrac{1}{2} e^*_{ij}. \tag{22}$$

Substituting (18) and (19) into (4) we obtain

$$\begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & pE_{33} & pE_{34} \\ E_{41} & E_{42} & pE_{43} & pE_{44} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{bmatrix} = \frac{a^2}{2\mu_1} \begin{bmatrix} 0 \\ 0 \\ f_{3n} \\ f_{4n} \end{bmatrix}, \tag{23}$$

where the matrix E_{ij} is the same as the one in (14) and

$$f_{30} = -\tfrac{1}{3}(2\sigma^*_{11} + \sigma^*_{33}),$$

$$f_{32} = \tfrac{2}{3}(\sigma^*_{11} - \sigma^*_{33}),$$

$$f_{3n} = 0, \quad \text{if } n \neq 0,2,$$

$$f_{42} = \tfrac{1}{3}(\sigma^*_{11} - \sigma^*_{33}),$$

$$f_{4n} = 0, \quad \text{if } n \neq 2. \tag{24}$$

It is seen from Eqs. (23) and (24) that A_n, B_n, C_n, and D_n $(n=0$ and $2)$ are the only nonvanishing expansion coefficients.

Let us now consider a special type of eigenstrain:

$$e^*_{ij} = e^* \delta_{ij}. \tag{25}$$

This corresponds to the uniform expansion (or contraction) of the dynamic inhomogeneity. Substituting (25) into (24) through (22), we have

$$f_{30} = -(2\mu_2 + 3\lambda_2)a^*,$$
$$f_{3n} = 0, \quad \forall n \in \mathbf{Z}^+ = \{1,2,3,...\}$$
$$f_{4n} = 0, \quad \forall n \in \mathbf{Z}^+ + \{0\}, \tag{26}$$

where

$$a^* = \tfrac{1}{2}e^*. \tag{27}$$

Solving Eq. (23) with (26), and substituting the results into (18) and (19), we finally obtain in Cartesian coordinates:

$$u_i^1(\mathbf{x}) = R(x_i/r)j_1(k_{L2}a)h_1(k_{L1}r), \quad \mathbf{x} \in \mathbf{R}^3 - \Omega,$$
$$u_i^2(\mathbf{x}) = R(x_i/r)j_1(k_{L2}r)h_1(k_{L1}a), \quad \mathbf{x} \in \Omega, \tag{28}$$

where

$$R = (2\mu_2 + 3\lambda_2/2S)k_{L1}k_{L2}a^3e^*,$$
$$S = \mu_2/2(k_{T2}a)^2k_{L1}aj_0(k_{L2}a)h_1(k_{L1}a)$$
$$-\mu_1/2(k_{T1}a)^2k_{L2}aj_1(k_{L2}a)h_0(k_{L1}a)$$
$$+2k_{L1}k_{L2}a^2j_1(k_{L2}a)h_1(k_{L1}a)(\mu_1 - \mu_2). \tag{29}$$

When materials 1 and 2 are the same, we have a dynamic inclusion problem.[14] Then we have

$$\mu_1 = \mu_2 = \mu, \quad \lambda_1 = \lambda_2 = \lambda. \tag{30}$$

Substituting (30) into (28) and (29), we obtain

$$u_i^1(\mathbf{x}) = R(x_i/r)j_1(k_La)h_1(k_Lr), \quad \mathbf{x} \in \mathbf{R}^3 - \Omega$$
$$u_i^2(\mathbf{x}) = R(x_i/r)j_1(k_Lr)h_1(k_La), \quad \mathbf{x} \in \Omega, \tag{31}$$

where

$$R = i(2\mu + 3\lambda/2\mu + \lambda)k_La^2e^*. \tag{32}$$

It can be shown that the displacement field given by Eqs. (31) and (32) is exactly the same as the one derived from the results for the dynamic inclusion problem.[14]

V. ESHELBY'S PROBLEM

The inhomogeneity problem with the static part of the eigenstrain $e_{ij}^*(t)$ given by (1) constitutes Eshelby's problem.[16,17] The elastic field inside the inhomogeneity can be obtained in a fairly simple manner by using Eshelby's equivalent inclusion method. The elastic field outside the inhomogeneity, however, is rather complicated even if we used Eshelby's result.[17] Here we will use the result obtained by Mikata and Nemat-Nasser[14] for computing the elastic field outside the inhomogeneity.

Following Eshelby, the strain for the inside of the inhomogeneity is given by

$$e_{ij} = S_{ijkl}(a_{kl}^* + e_{kl}^{**}) \quad \text{in } \Omega, \tag{33}$$

where S_{ijkl} is Eshelby's tensor[18] given by

$$S_{ijkl} = \frac{5\nu_1 - 1}{15(1-\nu_1)}\delta_{ij}\delta_{kl} + \frac{4-5\nu_1}{15(1-\nu_1)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{34}$$

and a_{kl}^* is the static part of the eigenstrain $e_{ij}^*(t)$ in (1), which is

$$a_{kl}^* = \tfrac{1}{2}e_{kl}^*. \tag{35}$$

e_{kl}^{**} in (33) is the fictitious eigenstrain, which is unknown at this stage. The equivalence relation between the inhomogeneity and the fictitious inclusion is given by

$$C_{ijkl}^2(e_{kl} - a_{kl}^*) = C_{ijkl}^1(e_{kl} - a_{kl}^* - e_{kl}^{**}) \quad \text{in } \Omega, \tag{36}$$

Equations (33) and (36) can be solved for the fictitious eigenstrain e_{kl}^{**}. After some algebraic manipulation, we obtain

$$2\Delta M_1 e_{ij}^{**} + \Delta L_1 e_{kk}^{**}\delta_{ij} = f_{ij}, \tag{37}$$

where

$$f_{ij} = R_{ijkl}a_{kl}^*,$$
$$R_{ijkl} = \Delta M_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \Delta L_2\delta_{ij}\delta_{kl},$$
$$\Delta M_1 = \mu_1 - M, \quad \Delta L_1 = \lambda_1 - L,$$
$$\Delta M_2 = \mu_2 - \mu_1 + M, \quad \Delta L_2 = \lambda_2 - \lambda_1 + L,$$
$$L = (2\mu + 3\lambda)\delta + 2\lambda\epsilon,$$
$$M = 2\mu\epsilon,$$
$$\mu = \mu_1 - \mu_2, \quad \lambda = \lambda_1 - \lambda_2,$$
$$\delta = \frac{5\nu_1 - 1}{15(1-\nu_1)}, \quad \epsilon = \frac{4 - 5\nu_1}{15(1-\nu_1)}. \tag{38}$$

Using (37), we have

$$e_{12}^{**} = e_{21}^{**} = \frac{f_{12}}{2\Delta M_1}, \quad e_{23}^{**} = e_{32}^{**} = \frac{f_{23}}{2\Delta M_1},$$
$$e_{31}^{**} = e_{13}^{**} = \frac{f_{31}}{2\Delta M_1}, \tag{39}$$

and

$$\begin{bmatrix} 2\Delta M_1 + \Delta L_1 & \Delta L_1 & \Delta L_1 \\ \Delta L_1 & 2\Delta M_1 + \Delta L_1 & \Delta L_1 \\ \Delta L_1 & \Delta L_1 & 2\Delta M_1 + \Delta L_1 \end{bmatrix} \begin{bmatrix} e_{11}^{**} \\ e_{22}^{**} \\ e_{33}^{**} \end{bmatrix} = \begin{bmatrix} f_{11} \\ f_{22} \\ f_{33} \end{bmatrix}. \tag{40}$$

No assumption has been made for the eigenstrain a_{ij}^* yet. If $a_{11}^* = a_{22}^* = a_{33}^* = a^*$, $a_{ij}^* = 0$ $(i \neq j)$, Eqs. (39) and (40) reduce to $e_{ij}^* = 0$ $(i \neq j)$ and

$$(2\Delta M_1 + 3\Delta L_1)e^{**} = f_{11}, \tag{41}$$

where $e^{**} = e^{**}_{11} = e^{**}_{22} = e^{**}_{33}$. Once the fictitious eigenstrain e^{**} is determined from (41), the stress field inside the inhomogeneity is obtained through (3b), (33), and (36) as

$$\sigma_{ij} = -\tfrac{4}{3} b \mu_1 (1+\nu_1/1-\nu_1)\delta_{ij} \quad \mathbf{x}\in\Omega, \tag{42}$$

where

$$b = a^*(2\mu_2 + 3\lambda_2/2\Delta M_1 + 3\Delta L_1). \tag{43}$$

Similarly, the strain field outside the inhomogeneity is given by

$$e_{ij} = S^{\text{out}}_{ijkl}(a^*_{kl} + e^{**}_{kl}) \quad \text{in } \mathbf{R}^3 - \Omega, \tag{44}$$

where S^{out}_{ijkl} is the static Eshelby tensor for the outside of the spherical inclusion obtained in Ref. 14 and given by

$$S^{\text{out}}_{ijkl} = \frac{a^3}{6}[D_{ijkl} + p(-A_{ijkl}\alpha + 3B_{ijkl}\beta - 3C_{ijkl}\gamma) + q\delta_{kl}(-A_{ijmm}\alpha + 3B_{ijmm}\beta - 3C_{ijmm}\gamma)], \tag{45}$$

where

$$A_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk},$$

$$B_{ijkl} = x_i x_j \delta_{kl} + x_j x_k \delta_{li} + x_k x_l \delta_{ij} + x_l x_i \delta_{jk} + x_i x_k \delta_{jl} + x_j x_l \delta_{jk},$$

$$C_{ijkl} = x_i x_j x_k x_l,$$

$$D_{ijkl} = \frac{2}{x^3}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + r\delta_{ij}\delta_{kl}) - \frac{3}{x^5}(x_i x_k \delta_{jl} + x_i x_l \delta_{jk} + x_j x_k \delta_{il} + x_j x_l \delta_{ik} + 2r x_i x_j \delta_{kl}),$$

$$\alpha = \frac{1}{x^3} - \frac{3a^2}{5x^5}, \tag{46}$$

$$\beta = \frac{1}{x^5} - \frac{a^2}{x^7},$$

$$\gamma = \frac{5}{x^7} - \frac{7a^2}{x^9},$$

$$p = \frac{1}{1-\nu_1},$$

$$q = \frac{\nu_1}{(1-\nu_1)(1-2\nu_1)},$$

$$r = \frac{2\nu_1}{1-2\nu_1}.$$

From (2b) and (44), the stress field outside the inhomogeneity is given by

$$\sigma_{ij} = C^1_{ijkl} S^{\text{out}}_{klmn}(a^*_{mn} + e^{**}_{mn}). \tag{47}$$

So far, we have made no assumption for the eigenstrain a^*_{ij}. Let us now consider the following eigenstrain:

$$a^*_{11} = a^*_{22} = a^*_{33} = a^*, \quad a^*_{ij} = 0 \quad (i \neq j). \tag{48}$$

Then we also have

$$e^{**}_{11} = e^{**}_{22} = e^{**}_{33}, \quad e^{**}_{ij} = 0 \quad (i \neq j). \tag{49}$$

TABLE I. Material properties.

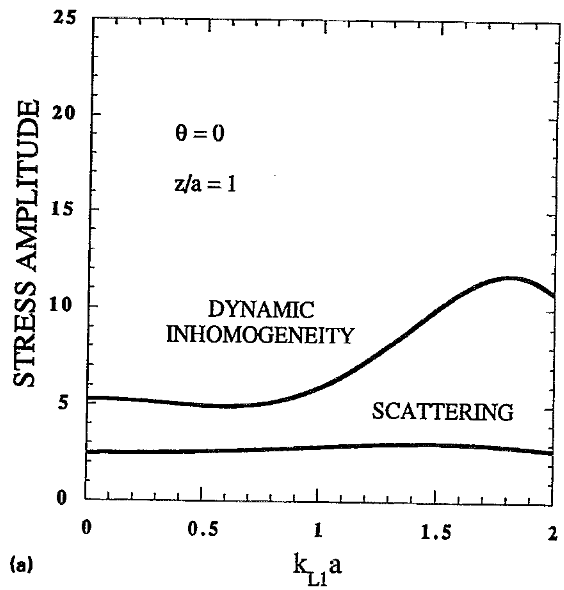
	E (GPa)	ρ (g/cm^3)	ν
Al_2O_3	380	3.7	0.22
t-ZrO_2	230	5.8	0.32

Thus we can write

$$a^*_{ij} + e^{**}_{ij} = b\delta_{ij}, \tag{50}$$

where b is given by Eq. (43). Substituting (50) into (47) and using (45) and (46), after a lengthy but straightforward calculation, we arrive at

$$\sigma_{ij} = \frac{2}{3} b\mu_1 \frac{1+\nu_1}{1-\nu_1}\left(\frac{a}{x}\right)^3\left(\delta_{ij} - 3\frac{x_i x_j}{x^2}\right), \quad \mathbf{x}\in\mathbf{R}^3 - \Omega, \tag{51}$$



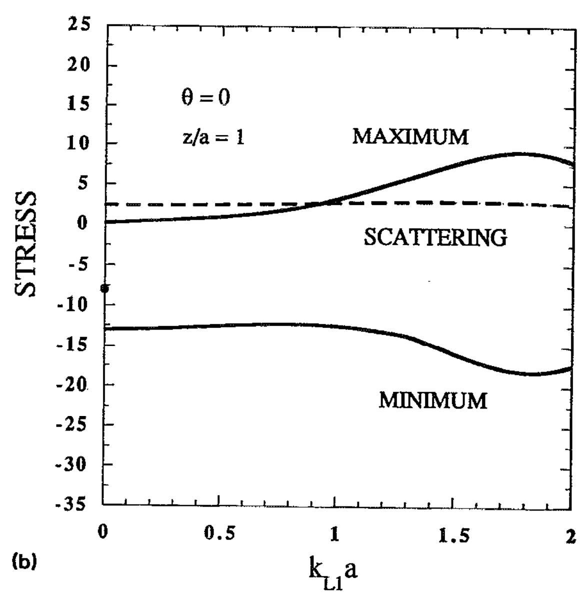


FIG. 3.(a) Stress (σ_{rr}) amplitude variations with $k_{L1}a$ at (0,0,a). (b) Combined stress (σ_{rr}) variation with $k_{L1}a$ at (0,0,a).

where

$$x^2 = x_k x_k. \tag{52}$$

VI. RESULTS AND DISCUSSION

In the previous sections, the three subproblems have been solved individually. Let us now consider the combined solution. For example, the combined stress field is given by

$$\sigma_{ij}(\mathbf{x},t) = \sigma_{ij}^{\mathrm{st}}(\mathbf{x}) + \sigma_{ij}^{\mathrm{sc}}(\mathbf{x})e^{-i\omega t} + \sigma_{ij}^{d}(\mathbf{x})e^{-2i\omega t}, \tag{53}$$

where $\sigma_{ij}^{\mathrm{st}}$, $\sigma_{ij}^{\mathrm{sc}}$, and σ_{ij}^{d} are the solutions of static inhomogeneity, scattering, and dynamic inhomogeneity, respectively. Only the real part of $\sigma_{ij}(\mathbf{x},t)$ is physically meaningful.

The material properties used for our model are given in Table I.[19,20] The matrix (region 1) is alumina and the inhomogeneity (region 2) is tetragonal zirconia. The tra[...] formation strain is modeled as

$$e_{11}^* = e_{22}^* = e_{33}^* (= e^*), \quad e_{ij}^* = 0 \quad (i \neq j), \qquad (\ldots$$

which gives a uniform expansion of the zirconia parti[...] The volumetric expansion of the zirconia particle is usu[...] expected as 3–5%,[7] which gives the transformation st[...] e^* as 1–1.7%. The strain component e_{33} of the incid[...] longitudinal wave may be, in most cases, 0.2–0.4%. T[...] in the following computations, the strain ratio e^*/e_{33} [...] taken as 5. In all of the following results, the stres[...] nondimensionalized by $\mu_1 e_{33}$. In Fig. 3(a), the st[...] (σ_{rr}) amplitude variations at (0,0,a) with a nondim[...] sional wave number $k_{L1}a$ for both scattering and dyna[...] inhomogeneity are shown. It is seen from Fig. 3(a) that [...] the high-frequency range, the stress amplitude for dyna[...] inhomogeneity becomes large. In Fig. 3(b), the maxim[...] and minimum of the combined stress (σ_{rr}) at (0,0,a)

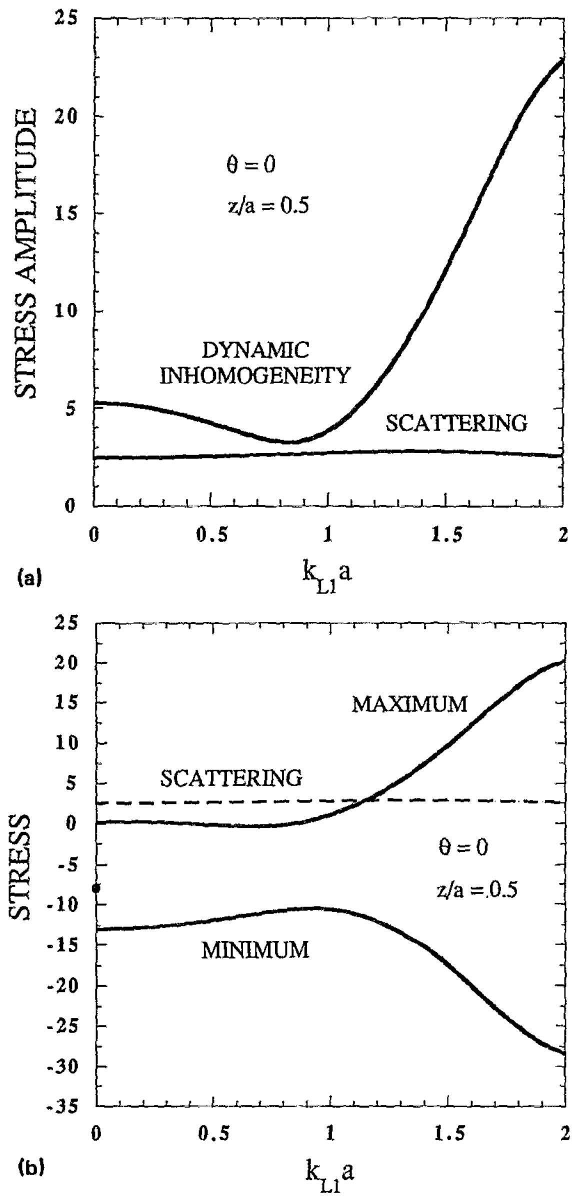


FIG. 4. (a) Stress (σ_{rr}) amplitude variations with $k_{L1}a$ at $(0,0,a/2)$. (b) Combined stress (σ_{rr}) variation with $k_{L1}a$ at $(0,0,a/2)$.

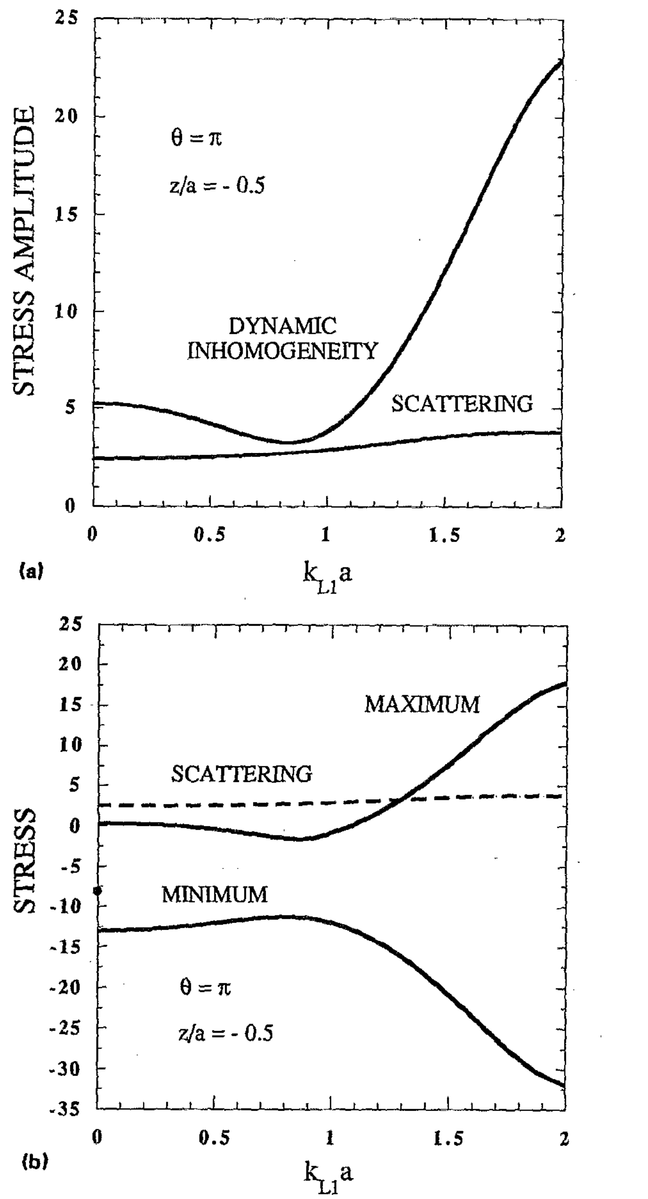


FIG. 5. (a) Stress (σ_{rr}) amplitude variations with $k_{L1}a$ at $(0,0,-a$[...] (b) Combined stress (σ_{rr}) variation with $k_{L1}a$ at $(0,0,-a/2)$.

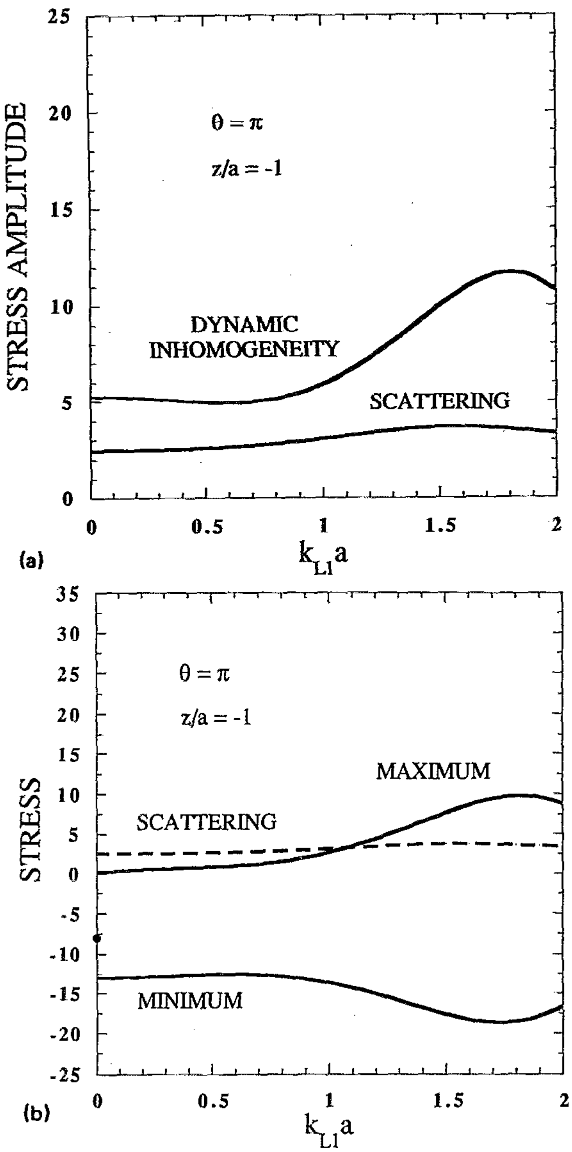


FIG. 6. Stress (σ_{rr}) amplitude variations with $k_{L1}a$ at $(0,0,-a)$. (b) Combined stress (σ_{rr}) variation with $k_{L1}a$ at $(0,0,-a)$.

shown for various nondimensional wave numbers $k_{L1}a$. The closed circle at $k_{L1}a = 0$ corresponds to the stress (σ_{rr}) in the static limit. As a reference, the stress amplitude for scattering is also shown in a dashed line. It is found from Fig. 3(b) that, in the high-frequency range, the maximum combined stress is larger than the maximum stress caused by scattering alone. Similar results are obtained in Figs. 4 through 6 for the points $(0,0,a/2)$, $(0,0,-a/2)$, and $(0,0,-a)$, respectively. It is naturally expected that the circumferential stress component ($\sigma_{\phi\phi}$) just outside the inhomogeneity becomes tensile when the inhomogeneity dilatationally expands. It should be emphasized, however, that we are concerned here with the radial stress components (σ_{rr}) at various locations along the z axis.

VII. CONCLUSION

An exact closed form solution has been found for a dynamic inhomogeneity problem. By using this exact solution, the interaction between the incident harmonic stress wave and the dynamically transforming spherical inhomogeneity, which serves as a model to investigate the elastodynamic behavior of a transformation-toughened ceramics, has been studied. Maximum and minimum stress variations with the incident wave frequency have been obtained within the transforming particle. It has been found that, in the high-frequency range, the dynamic transformation makes the maximum combined stress larger than the tensile stress which would have been obtained by the incident harmonic stress wave alone.

It should be emphasized here that, in principle by using the present results, any form of an incident stress pulse in time and any form of a transformation strain in time can be handled by way of Fourier decomposition.

ACKNOWLEDGMENT

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